

2.4 Exercices

Exercise 2.1.

Consider the following sets: $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$ and

$C = \{2, 4, 6, 8, 10\}$. Find the following sets: $(A \cup B)$, $(B \cap C)$, $(A - B)$ and $(A \Delta C)$.

Exercise 2.2.

Let $E = \{a, b, c\}$ be a set. Can we write:)

1) $a \in E$, 2) $a \subset E$, 3) $\{a\} \subset E$, 4) $\emptyset \in E$, 5) $\emptyset \subset E$, 6) $\{\emptyset\} \subset E$?

Exercise 2.3.

Let A , B , and C be three subsets of a set E . Prove that:

1. $A \setminus B = A \cap B^c$.
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
4. $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Exercise 2.4. (Power set)

Let $E = \{a, b, c, d\}$. Find the power set $\mathcal{P}(E)$ of E , which is the set of all subsets of E . Give an example of a partition of E , which is a collection of non-empty disjoint subsets of E whose union equals E .

Exercise 2.5.

1. What are the images of the sets $\mathbb{R}$, $[0, 2\pi]$, $[0, \frac{\pi}{2}]$, and the inverse image of the sets $[0, 1]$, $[3, 4]$, $[1, 2]$ under the function $f(x) = \sin(x)$?
2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2 + 1$. Consider the sets $A = [-3, 2]$ and $B = [0, 4]$. Compare $f(A \setminus B)$ and $f(A) \setminus f(B)$.
3. What condition must function f satisfy so that $f(A \setminus B) = f(A) \setminus f(B)$?

Exercise 2.6.

Determine whether the following relations are reflexive, symmetric, antisymmetric, and transitive:

1. $E = \mathbb{Z}$ and $x \mathcal{R} y \Leftrightarrow |x| = |y|$.

2. $E = \mathbb{R} \setminus \{0\}$ and $x\mathcal{R}y \Leftrightarrow xy > 0$.
3. $E = \mathbb{Z}$ and $x\mathcal{R}y \Leftrightarrow x - y$ is even.

Identify among the above examples which relations are orders and which are equivalence relations.

Exercise 2.7.

Determine whether the following relations are reflexive, symmetric, antisymmetric, and transitive:

1. $E = \mathbb{R}$ and $x\mathcal{R}y \Leftrightarrow x = -y$.
2. $E = \mathbb{R}$ and $x\mathcal{R}y \Leftrightarrow \cos^2(x) + \sin^2(y) = 1$.
3. $E = \mathbb{N}$ and $x\mathcal{R}y \Leftrightarrow \exists p, q \geq 1$ such that $y = px^q$ (where p and q are integers).

Identify among the above examples which relations are orders and which are equivalence relations.

Exercise 2.8.

Let $E = \mathbb{Z}$, the set of all integers. Consider the following relations on E :

1. Relation $\sim_1$ defined by $x \sim_1 y$ if and only if $x + y$ is even.
2. Relation $\sim_2$ defined by $x \sim_2 y$ if and only if x and y have the same remainder when divided by 5.
3. Relation $\sim_3$ defined by $x \sim_3 y$ if and only if $x - y$ is a multiple of 7.

For each relation $\sim_i$ (where $i = 1, 2, 3$):

1. Determine if $\sim_i$ is an equivalence relation on $\mathbb{Z}$. Explain why or why not.
2. If $\sim_i$ is an equivalence relation, identify the equivalence classes of $\mathbb{Z}$ under $\sim_i$.

Exercise 2.9.

Let $\mathcal{R}$ be an equivalence relation on a non-empty set E . Show that

$$\forall x, y \in E, \quad x\mathcal{R}y \quad \Leftrightarrow \quad \dot{x} = \dot{y}.$$

Exercise 2.10.

Let $\mathbb{N}^*$ denote the set of positive integers. Define the relation $\mathcal{R}$ on $\mathbb{N}^*$ by $x\mathcal{R}y$ if and only if x divides y .

1. Show that $\mathcal{R}$ is a partial order relation on $\mathbb{N}^*$.
2. Is $\mathcal{R}$ a total order relation?
3. Describe the sets $\{x \in \mathbb{N}^* \mid x\mathcal{R}5\}$ and $\{x \in \mathbb{N}^* \mid 5\mathcal{R}x\}$.
4. Does $\mathbb{N}^*$ have a least element? A greatest element?

Exercise 2.11.

Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x^2 + x - 2$.

1. Give the definition of $f^{-1}(\{4\})$. Calculate $f^{-1}(\{4\})$.
2. Is the function f bijective?
3. Give the definition of $f([-1, 1])$. Calculate $f([-1, 1])$.
4. Give the definition of $f^{-1}([-2, 4])$. Calculate $f^{-1}([-2, 4])$.

Exercise 2.12.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{2x}{1+x^2}$.

1. Is f injective? Is f surjective?
2. Show that $f(\mathbb{R}) = [-1, 1]$.
3. Show that the restriction $g : [-1, 1] \longrightarrow [-1, 1]$ defined by $g(x) = f(x)$ is a bijection.

Exercise 2.13.

Let $f : E \rightarrow F$, $g : F \rightarrow G$, and $h = g \circ f$.

1. Show that if h is injective, then f is injective. Also, show that if h is surjective, then g is surjective.
2. Show that if h is surjective and g is injective, then f is surjective.
3. Show that if h is injective and f is surjective, then g is injective.

Exercise 2.14.

Let E be a set, and A and B be two subsets of E . Prove the following properties:

1. $\phi_A + \phi_{A^c} = 1$, where $A^c = E \setminus A$.

2. $\phi_{A \cap B} = \phi_A \cdot \phi_B$.

3. $\phi_{A \cup B} = \phi_A + \phi_B - \phi_A \cdot \phi_B$.

4. $\phi_{A \setminus B} = \phi_A(1 - \phi_B)$.

where ϕ_A is the indicator function of A , defined as

$$\phi_A : E \longrightarrow \{0, 1\}, \quad x \longmapsto \phi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$