

Tutorial Exercises 1

Exercise 1 – Estimating a Transition Matrix from Data

The load of a computer server is recorded once per minute, giving the following sequence of 13 observations:

$L, L, M, H, M, M, L, L, M, H, H, M, L$

where

$L = \text{Low}, \quad M = \text{Medium}, \quad H = \text{High}.$

Assume that the load follows a Markov chain.

1. Define the state space E .
2. For each state, count the transitions to every state and present the counts in a table.
3. Build the transition matrix $\hat{P}$.

Exercise 2 – Representing a Markov Chain as a Transition Graph

A website has three pages,

$$E = \{A, B, C\}.$$

A user navigates as follows:

- From A , the user goes to B or to C , with probability 0.5 each.
 - From B , the user always goes to C .
 - From C , the user always goes back to A .
1. Write the transition matrix P .
 2. Draw the directed transition graph of the chain.
 3. Compute P^2 .
 4. Starting from page A , what is the probability of being on page A after two clicks?
 5. List the possible two-step paths from A back to A and give the probability of each path.
 6. Verify your answer to Question 4 using the Chapman–Kolmogorov equation.

Exercise 3 – Stationary Distribution

Let (X_n) be a Markov chain with two states, 1 and 2, such that

$$p_{12} = 0.25, \quad p_{21} = 0.5.$$

1. Write the transition matrix P and draw the corresponding state-transition graph.
2. Calculate P^3 and deduce the transition probability from state 1 to state 2 in three steps,

$$p_{12}^{(3)}.$$

3. Verify your answer to Question 2 using the Chapman–Kolmogorov equation.
4. Given the initial distribution

$$\pi_0 = (0.2, 0.8),$$

Calculate the distributions

$$\pi_1, \quad \pi_2, \quad \pi_3,$$

5. Determine the stationary distribution π .

Exercise 4 – Communicating Classes, Recurrence, and Periodicity

Consider the Markov chain with transition matrix

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0.2 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

1. Draw the transition graph.
2. Determine which states communicate with each other.
3. Identify all communicating classes.
4. Determine which communicating classes are closed.
5. Classify the states as recurrent or transient.
6. Determine the period of each class.
7. Find all stationary distributions of the chain.
8. Explain why the stationary distribution is concentrated on the closed recurrent class.

Exercise 5 – Modeling the Spread of a Disease

Consider a population in which individuals may contract a disease. An individual can be in one of the following three states: $E = \{\text{Susceptible}, \text{Sick}, \text{Immune}\}$. A **Susceptible** individual is healthy but not immune, a **Sick** individual has the disease, and an **Immune** individual is healthy and partially protected against the disease..

The state of an individual may change from one day to the next according to the following rules:

- A Susceptible individual remains Susceptible with probability 0.95 and becomes Sick with probability 0.05.
- A Sick individual remains Sick with probability 0.90 and becomes Immune with probability 0.10.
- An Immune individual remains Immune with probability 0.98 and becomes Susceptible with probability 0.02.

Assume that the evolution of an individual's health status can be modeled as a discrete-time Markov chain, with one time step corresponding to one day.

1. Construct the transition matrix P of the corresponding Markov chain where S, M, and I denote Susceptible, Sick, and Immune, respectively. Draw the corresponding directed transition graph.
2. Initially, 4% of the population is Sick and the remaining 96% is Susceptible. No individual is initially Immune.

Calculate the proportion of the population in each state after 30 days.

To facilitate the calculations, use the following approximation:

$$P^{15} = \begin{pmatrix} 0.487 & 0.261 & 0.253 \\ 0.101 & 0.226 & 0.673 \\ 0.185 & 0.051 & 0.764 \end{pmatrix}.$$

3. Determine whether the Markov chain is **irreducible**, **aperiodic**, and **recurrent**. Justify each answer.
4. Determine whether the chain admits a **unique stationary distribution**. Justify your answer and compute the stationary distribution.
5. A computer calculation of the 100th power of the transition matrix gives the following result, rounded to three decimal places:

$$P^{100} = \begin{pmatrix} 0.250 & 0.125 & 0.625 \\ 0.250 & 0.125 & 0.625 \\ 0.250 & 0.125 & 0.625 \end{pmatrix}.$$

Suppose that the population consists of 50 million individuals. What will be the approximate number of individuals in each state in the long run?

University of Mila

Faculty of Mathematics and Computer Science

Department of Computer Science

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Tutorial Exercises 1 – Solutions**Exercise 1 – Estimating a Transition Matrix from Data**

The observed sequence is

$$L, L, M, H, M, M, L, L, M, H, H, M, L.$$

We have

$$L = \text{Low}, \quad M = \text{Medium}, \quad H = \text{High}.$$

1. State space

The possible server-load states are

$$E = \{L, M, H\}.$$

2. Counting the transitions

We consider each pair of consecutive observations:

$$\begin{array}{cccccc} L \rightarrow L & L \rightarrow M & M \rightarrow H & H \rightarrow M & M \rightarrow M & M \rightarrow L \\ L \rightarrow L & L \rightarrow M & M \rightarrow H & H \rightarrow H & H \rightarrow M & M \rightarrow L \end{array}$$

The transition counts are therefore

$$N = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

The rows correspond to the current state L, M, H , and the columns correspond to the next state L, M, H .

Thus,

	L	M	H	Total
L	2	2	0	4
M	2	1	2	5
H	0	2	1	3

There are 12 transitions because 13 observations produce $13 - 1 = 12$ consecutive transitions.**3. Empirical transition matrix**

The empirical transition probability is

$$\hat{p}_{ij} = \frac{N_{ij}}{\sum_k N_{ik}}.$$

Therefore,

$$\hat{P} = \begin{pmatrix} \frac{2}{4} & \frac{2}{4} & \frac{0}{4} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{0}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Hence,

$$\hat{P} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.4 & 0.2 & 0.4 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Each row sums to 1, as required:

$$0.5 + 0.5 + 0 = 1,$$

$$0.4 + 0.2 + 0.4 = 1,$$

$$0 + \frac{2}{3} + \frac{1}{3} = 1.$$

Exercise 2 – Representing a Markov Chain as a Transition Graph

The state space is

$$E = \{A, B, C\}.$$

The transition probabilities are

$$p_{AB} = 0.5, \quad p_{AC} = 0.5,$$

$$p_{BC} = 1, \quad p_{CA} = 1.$$

All other transition probabilities are zero.

1. Transition matrix

Using

$$p_{ij} = P(X_{n+1} = j \mid X_n = i),$$

the transition matrix is

$$P = \begin{pmatrix} p_{AA} & p_{AB} & p_{AC} \\ p_{BA} & p_{BB} & p_{BC} \\ p_{CA} & p_{CB} & p_{CC} \end{pmatrix}.$$

For state A ,

$$p_{AA} + p_{AB} + p_{AC} = 1.$$

Thus,

$$p_{AA} = 1 - p_{AB} - p_{AC} = 1 - 0.5 - 0.5 = 0.$$

For state B ,

$$p_{BA} + p_{BB} + p_{BC} = 1.$$

Since $p_{BC} = 1$,

$$p_{BA} + p_{BB} = 0,$$

and therefore

$$p_{BA} = p_{BB} = 0.$$

For state C ,

$$p_{CA} + p_{CB} + p_{CC} = 1.$$

Since $p_{CA} = 1$,

$$p_{CB} + p_{CC} = 0,$$

and therefore

$$p_{CB} = p_{CC} = 0.$$

Hence,

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

2. Transition graph

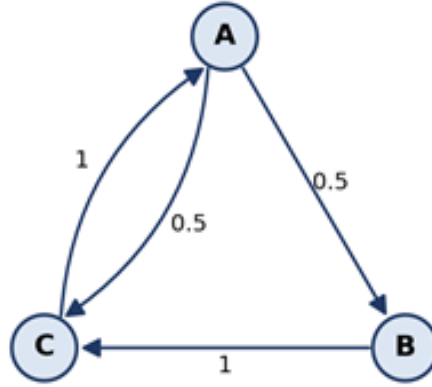


Figure 1: Transition graph of the Markov chain.

3. Calculation of P^2

We compute

$$P^2 = P \times P.$$

Therefore,

$$P^2 = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The entries are

$$(P^2)_{11} = 0(0) + 0.5(0) + 0.5(1) = 0.5,$$

$$(P^2)_{12} = 0(0.5) + 0.5(0) + 0.5(0) = 0,$$

$$(P^2)_{13} = 0(0.5) + 0.5(1) + 0.5(0) = 0.5.$$

Similarly,

$$(P^2)_{21} = 1, \quad (P^2)_{22} = 0, \quad (P^2)_{23} = 0,$$

and

$$(P^2)_{31} = 0, \quad (P^2)_{32} = 0.5, \quad (P^2)_{33} = 0.5.$$

Thus,

$$P^2 = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

4. Probability of being on page A after two clicks

Starting from A ,

$$P(X_2 = A \mid X_0 = A) = (P^2)_{AA}.$$

From P^2 ,

$$(P^2)_{AA} = 0.5.$$

Therefore,

$$P(X_2 = A \mid X_0 = A) = 0.5$$

5. Two-step paths from A to A

The possible intermediate states are B and C .

For the path

$$A \rightarrow B \rightarrow A,$$

the probability is

$$P(A \rightarrow B \rightarrow A) = p_{AB}p_{BA} = 0.5 \times 0 = 0.$$

Thus, this path is impossible.

For the path

$$A \rightarrow C \rightarrow A,$$

the probability is

$$P(A \rightarrow C \rightarrow A) = p_{AC}p_{CA} = 0.5 \times 1 = 0.5.$$

Therefore, the only possible two-step return path is

$$A \rightarrow C \rightarrow A$$

with probability

$$0.5.$$

6. Verification using Chapman–Kolmogorov

The Chapman–Kolmogorov equation gives

$$p_{AA}^{(2)} = \sum_{k \in E} p_{Ak}p_{kA}.$$

Since $E = \{A, B, C\}$,

$$p_{AA}^{(2)} = p_{AA}p_{AA} + p_{AB}p_{BA} + p_{AC}p_{CA}.$$

Substituting the values,

$$p_{AA}^{(2)} = (0)(0) + (0.5)(0) + (0.5)(1).$$

Hence,

$$p_{AA}^{(2)} = 0.5$$

which confirms the result obtained from P^2 .

Exercise 3 – Stationary Distribution

Let

$$p_{12} = 0.25, \quad p_{21} = 0.5.$$

Because the rows of a transition matrix must sum to 1,

$$p_{11} = 1 - p_{12} = 0.75,$$

and

$$p_{22} = 1 - p_{21} = 0.5.$$

1. Transition matrix and state-transition graph

The transition matrix is

$$P = \begin{pmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{pmatrix}$$

The corresponding transition graph can be inserted using:

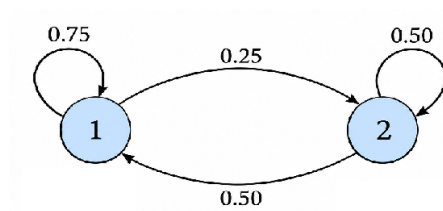


Figure 2: Transition graph of the Markov chain.

2. Calculation of P^3

First,

$$P^2 = P \times P.$$

Thus,

$$P^2 = \begin{pmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{pmatrix} \begin{pmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{pmatrix}.$$

Therefore,

$$P^2 = \begin{pmatrix} 0.75(0.75) + 0.25(0.50) & 0.75(0.25) + 0.25(0.50) \\ 0.50(0.75) + 0.50(0.50) & 0.50(0.25) + 0.50(0.50) \end{pmatrix}.$$

Hence,

$$P^2 = \begin{pmatrix} 0.6875 & 0.3125 \\ 0.625 & 0.375 \end{pmatrix}.$$

Now,

$$P^3 = P^2 P.$$

Thus,

$$P^3 = \begin{pmatrix} 0.6875 & 0.3125 \\ 0.625 & 0.375 \end{pmatrix} \begin{pmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{pmatrix}.$$

Therefore,

$$P^3 = \begin{pmatrix} 0.671875 & 0.328125 \\ 0.65625 & 0.34375 \end{pmatrix}$$

The three-step transition probability from state 1 to state 2 is

$$p_{12}^{(3)} = (P^3)_{12}.$$

Hence,

$$p_{12}^{(3)} = 0.328125$$

3. Verification using Chapman–Kolmogorov

Using the Chapman–Kolmogorov equation with $n = 2$ and $m = 1$,

$$p_{12}^{(3)} = \sum_{k=1}^2 p_{1k}^{(2)} p_{k2}.$$

Therefore,

$$p_{12}^{(3)} = p_{11}^{(2)} p_{12} + p_{12}^{(2)} p_{22}.$$

Therefore,

$$\begin{aligned} p_{12}^{(3)} &= (p_{11}p_{11} + p_{12}p_{21})p_{12} + (p_{11}p_{12} + p_{12}p_{22})p_{22} \\ &= p_{11}p_{11}p_{12} + p_{12}p_{21}p_{12} + p_{11}p_{12}p_{22} + p_{12}p_{22}p_{22}. \end{aligned}$$

Therefore,

$$\begin{aligned} p_{12}^{(3)} &= p_{11}p_{11}p_{12} + p_{11}p_{12}p_{22} + p_{12}p_{21}p_{12} + p_{12}p_{22}p_{22} \\ &= (0.75)(0.75)(0.25) + (0.75)(0.25)(0.50) + (0.25)(0.50)(0.25) + (0.25)(0.50)(0.50) \\ &= 0.140625 + 0.09375 + 0.03125 + 0.0625 \\ &= 0.328125. \end{aligned}$$

$$p_{12}^{(3)} = 0.328125$$

which confirms the result obtained from P^3 .

4. Distribution at times 1, 2, and 3

5. Solution

The initial distribution is

$$\pi_0 = (0.2, 0.8).$$

The distribution evolves according to

$$\pi_{n+1} = \pi_n P,$$

where

$$P = \begin{pmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{pmatrix}.$$

Step 1: Calculate π_1

$$\begin{aligned}
\pi_1 &= \pi_0 P \\
&= (0.2, 0.8) \begin{pmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{pmatrix} \\
&= (0.2(0.75) + 0.8(0.50), 0.2(0.25) + 0.8(0.50)) \\
&= (0.15 + 0.40, 0.05 + 0.40) \\
&= \boxed{(0.55, 0.45)}.
\end{aligned}$$

Step 2: Calculate π_2

$$\begin{aligned}
\pi_2 &= \pi_1 P \\
&= (0.55, 0.45) \begin{pmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{pmatrix} \\
&= (0.55(0.75) + 0.45(0.50), 0.55(0.25) + 0.45(0.50)) \\
&= (0.4125 + 0.225, 0.1375 + 0.225) \\
&= \boxed{(0.6375, 0.3625)}.
\end{aligned}$$

Step 3: Calculate π_3

$$\begin{aligned}
\pi_3 &= \pi_2 P \\
&= (0.6375, 0.3625) \begin{pmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{pmatrix} \\
&= (0.6375(0.75) + 0.3625(0.50), 0.6375(0.25) + 0.3625(0.50)) \\
&= (0.478125 + 0.18125, 0.159375 + 0.18125) \\
&= \boxed{(0.659375, 0.340625)}.
\end{aligned}$$

Therefore, the distributions are

$$\pi_1 = (0.55, 0.45), \quad \pi_2 = (0.6375, 0.3625), \quad \pi_3 = (0.659375, 0.340625).$$

6. Stationary distribution

A stationary distribution $\pi = (p_1, p_2)$ satisfies

$$\pi P = \pi,$$

together with

$$p_1 + p_2 = 1.$$

Therefore,

$$(p_1, p_2) \begin{pmatrix} 0.75 & 0.25 \\ 0.50 & 0.50 \end{pmatrix} = (p_1, p_2).$$

From the first component,

$$0.75p_1 + 0.50p_2 = p_1.$$

Thus,

$$0.50p_2 = 0.25p_1,$$

so

$$p_1 = 2p_2.$$

Using

$$p_1 + p_2 = 1,$$

we obtain

$$2p_2 + p_2 = 1,$$

hence

$$p_2 = \frac{1}{3}, \quad p_1 = \frac{2}{3}.$$

Therefore,

$$\pi = \left(\frac{2}{3}, \frac{1}{3} \right)$$

This is the long-term stationary distribution of the chain.

Exercise 4 – Communicating Classes, Recurrence, and Periodicity

Consider

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0.2 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

1. Transition graph

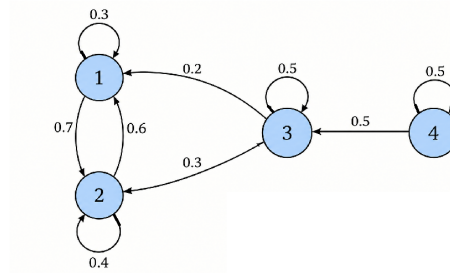


Figure 3: Transition graph of the Markov chain.

2. Communicating states States 1 and 2 communicate because

$$1 \longrightarrow 2$$

and

$$2 \longrightarrow 1.$$

Therefore,

$$1 \leftrightarrow 2.$$

State 3 can reach states 1 and 2, since

$$3 \longrightarrow 1$$

and

$$3 \longrightarrow 2.$$

However, neither state 1 nor state 2 can reach state 3. Thus,

$$3 \nrightarrow 1, \quad 3 \nrightarrow 2.$$

State 4 can reach state 3, and consequently states 1 and 2, but none of these states can return to 4. Therefore,

$$4 \nrightarrow 3, \quad 4 \nrightarrow 1, \quad 4 \nrightarrow 2.$$

3. Communicating classes

The communicating classes are

$$\{1, 2\}, \quad \{3\}, \quad \{4\}.$$

4. Closed communicating classes

A class is closed if no transition from a state in the class leads outside the class.

For the class $\{1, 2\}$, all transitions remain inside the class:

$$1 \rightarrow 1, \quad 1 \rightarrow 2, \quad 2 \rightarrow 1, \quad 2 \rightarrow 2.$$

Therefore,

$$\{1, 2\} \text{ is closed.}$$

For state 3,

$$3 \rightarrow 1, \quad 3 \rightarrow 2,$$

so $\{3\}$ is not closed.

For state 4,

$$4 \rightarrow 3,$$

so $\{4\}$ is not closed.

Hence, the only closed communicating class is

$$\{1, 2\}.$$

5. Classification of states

Because the chain is finite, every state belonging to a closed communicating class is recurrent.

Therefore,

$$1 \text{ and } 2 \text{ are recurrent.}$$

States 3 and 4 belong to non-closed classes and can leave their classes without any possibility of returning.

Therefore,

$$3 \text{ and } 4 \text{ are transient.}$$

Thus,

$$\text{Recurrent states} = \{1, 2\}$$

and

$$\text{Transient states} = \{3, 4\}.$$

6. Period of each class

The period of a state i is

$$d(i) = \gcd\{n \geq 1 : p_{ii}^{(n)} > 0\}.$$

State 1 has a self-loop because

$$p_{11} = 0.3 > 0.$$

Therefore,

$$d(1) = 1.$$

Similarly,

$$p_{22} = 0.4 > 0,$$

so

$$d(2) = 1.$$

Since communicating states have the same period,

$$\boxed{d(\{1, 2\}) = 1}.$$

State 3 has a self-loop:

$$p_{33} = 0.5 > 0,$$

so

$$\boxed{d(\{3\}) = 1}.$$

State 4 also has a self-loop:

$$p_{44} = 0.5 > 0,$$

so

$$\boxed{d(\{4\}) = 1}.$$

Thus, all communicating classes are aperiodic.

7. Stationary distributions

Let

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4).$$

A stationary distribution satisfies

$$\pi P = \pi$$

and

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1.$$

The equations are

$$\pi_1 = 0.3\pi_1 + 0.6\pi_2 + 0.2\pi_3,$$

$$\pi_2 = 0.7\pi_1 + 0.4\pi_2 + 0.3\pi_3,$$

$$\pi_3 = 0.5\pi_3 + 0.5\pi_4,$$

$$\pi_4 = 0.5\pi_4.$$

From the last equation,

$$\pi_4 = 0.$$

Then the third equation gives

$$\pi_3 = 0.$$

Thus, the stationary distribution is concentrated on the closed class $\{1, 2\}$.
The remaining equations become

$$\pi_1 = 0.3\pi_1 + 0.6\pi_2,$$

$$\pi_2 = 0.7\pi_1 + 0.4\pi_2,$$

with

$$\pi_1 + \pi_2 = 1.$$

From the first equation,

$$0.7\pi_1 = 0.6\pi_2,$$

so

$$\frac{\pi_1}{\pi_2} = \frac{6}{7}.$$

Therefore,

$$\pi_1 = \frac{6}{13}, \quad \pi_2 = \frac{7}{13}.$$

Hence, the unique stationary distribution is

$$\pi = \left(\frac{6}{13}, \frac{7}{13}, 0, 0 \right).$$

8. Why is the stationary distribution concentrated on the closed recurrent class?

States 3 and 4 are transient. Probability can move from these states toward the closed class

$$\{1, 2\},$$

but once the chain enters $\{1, 2\}$, it cannot leave it.

In particular,

$$3 \longrightarrow 1, 2$$

and

$$4 \longrightarrow 3 \longrightarrow 1, 2,$$

while there is no transition from 1 or 2 back to 3 or 4.

Therefore, in the long run, the probability mass cannot remain on the transient states. A stationary distribution must consequently assign

$$\pi_3 = \pi_4 = 0.$$

All stationary probability mass is concentrated on the unique closed recurrent class:

$$\{1, 2\}.$$

Thus,

$$\pi = \left(\frac{6}{13}, \frac{7}{13}, 0, 0 \right).$$

Exercise 5 – Modeling the Spread of a Disease

The state space is

$$E = \{S, M, I\},$$

where S denotes Susceptible, M denotes Sick, and I denotes Immune.

1. Transition matrix and transition graph

Using the state order

$$(S, M, I),$$

the given transition probabilities are

$$p_{SS} = 0.95, \quad p_{SM} = 0.05,$$

$$p_{MM} = 0.90, \quad p_{MI} = 0.10,$$

$$p_{II} = 0.98, \quad p_{IS} = 0.02.$$

All other transition probabilities are zero.

Therefore, the transition matrix is

$$P = \begin{pmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.90 & 0.10 \\ 0.02 & 0 & 0.98 \end{pmatrix}$$

Each row sums to 1:

$$0.95 + 0.05 + 0 = 1,$$

$$0 + 0.90 + 0.10 = 1,$$

$$0.02 + 0 + 0.98 = 1.$$

Hence, P is a stochastic matrix.

The transition graph can be inserted using:

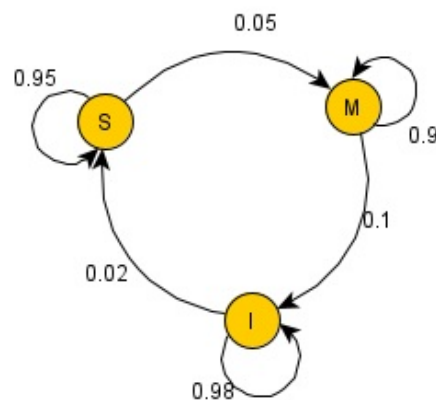


Figure 4: Transition graph of the disease-state Markov chain.

2. Distribution after 30 days

Initially, 4% of the population is Sick and the remaining 96% is Susceptible. No individual is initially Immune.

Thus, the initial distribution is

$$\pi_0 = (0.96, 0.04, 0)$$

with the state order

$$(S, M, I).$$

The distribution after n days is

$$\pi_n = \pi_0 P^n.$$

Since

$$P^{30} = P^{15} P^{15},$$

we can use the given approximation

$$P^{15} = \begin{pmatrix} 0.487 & 0.261 & 0.253 \\ 0.101 & 0.226 & 0.673 \\ 0.185 & 0.051 & 0.764 \end{pmatrix}.$$

First, calculate the distribution after 15 days:

$$\begin{aligned} \pi_{15} &= \pi_0 P^{15} \\ &= (0.96, 0.04, 0) \begin{pmatrix} 0.487 & 0.261 & 0.253 \\ 0.101 & 0.226 & 0.673 \\ 0.185 & 0.051 & 0.764 \end{pmatrix} \\ &= (0.96(0.487) + 0.04(0.101), \\ &\quad 0.96(0.261) + 0.04(0.226), \\ &\quad 0.96(0.253) + 0.04(0.673)) \\ &= (0.47156, 0.25960, 0.26980). \end{aligned}$$

Therefore,

$$\pi_{15} \approx (0.472, 0.260, 0.270)$$

Now calculate the distribution after 30 days:

$$\pi_{30} = \pi_{15} P^{15}.$$

Thus,

$$\pi_{30} = (0.47156, 0.25960, 0.26980) \begin{pmatrix} 0.487 & 0.261 & 0.253 \\ 0.101 & 0.226 & 0.673 \\ 0.185 & 0.051 & 0.764 \end{pmatrix}.$$

Hence,

$$\begin{aligned} \pi_{30,S} &= 0.47156(0.487) + 0.25960(0.101) + 0.26980(0.185) \\ &\approx 0.30578, \end{aligned}$$

$$\begin{aligned} \pi_{30,M} &= 0.47156(0.261) + 0.25960(0.226) + 0.26980(0.051) \\ &\approx 0.19551, \end{aligned}$$

and

$$\begin{aligned}\pi_{30,I} &= 0.47156(0.253) + 0.25960(0.673) + 0.26980(0.764) \\ &\approx 0.50014.\end{aligned}$$

Therefore,

$$\pi_{30} \approx (0.306, 0.196, 0.500)$$

Thus, after 30 days, approximately

$$30.6\% \text{ are Susceptible, } 19.6\% \text{ are Sick, } 50.0\% \text{ are Immune.}$$

The small numerical difference that may appear when using the original matrix P directly is due to the fact that the supplied matrix P^{15} is rounded to three decimal places.

3. Irreducibility, aperiodicity, and recurrence

Irreducibility:

The chain is irreducible if every state can be reached from every other state.

We have

$$S \rightarrow M, \quad M \rightarrow I, \quad I \rightarrow S,$$

with positive probabilities:

$$p_{SM} > 0, \quad p_{MI} > 0, \quad p_{IS} > 0.$$

Therefore,

$$S \rightarrow M \rightarrow I \rightarrow S,$$

so every state can reach every other state.

Hence,

$$\boxed{\text{The chain is irreducible.}}$$

Aperiodicity:

Each state has a positive self-transition probability:

$$p_{SS} = 0.95 > 0, \quad p_{MM} = 0.90 > 0, \quad p_{II} = 0.98 > 0.$$

Therefore, each state can return to itself in one step. Hence,

$$d(S) = d(M) = d(I) = 1.$$

Thus,

$$\boxed{\text{The chain is aperiodic.}}$$

Recurrence:

The state space is finite and the chain is irreducible. Therefore, all states are recurrent (in fact, positive recurrent).

Hence,

$$\boxed{S, M, I \text{ are all recurrent.}}$$

4. Stationary distribution

Because the chain is finite and irreducible, it admits a unique stationary distribution.

Let

$$\pi = (\pi_S, \pi_M, \pi_I).$$

We need to solve

$$\pi P = \pi$$

subject to

$$\pi_S + \pi_M + \pi_I = 1.$$

Thus,

$$(\pi_S, \pi_M, \pi_I) \begin{pmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.90 & 0.10 \\ 0.02 & 0 & 0.98 \end{pmatrix} = (\pi_S, \pi_M, \pi_I).$$

This gives

$$\begin{cases} 0.95\pi_S + 0.02\pi_I = \pi_S, \\ 0.05\pi_S + 0.90\pi_M = \pi_M, \\ 0.10\pi_M + 0.98\pi_I = \pi_I. \end{cases}$$

From the first equation,

$$0.05\pi_S = 0.02\pi_I,$$

so

$$\pi_I = 2.5\pi_S.$$

From the second equation,

$$0.05\pi_S = 0.10\pi_M,$$

so

$$\pi_M = 0.5\pi_S.$$

Using

$$\pi_S + \pi_M + \pi_I = 1,$$

we obtain

$$\pi_S + 0.5\pi_S + 2.5\pi_S = 1.$$

Therefore,

$$4\pi_S = 1,$$

and hence

$$\pi_S = \frac{1}{4}.$$

Consequently,

$$\pi_M = \frac{1}{8}, \quad \pi_I = \frac{5}{8}.$$

Thus, the unique stationary distribution is

$$\boxed{\pi = \left(\frac{1}{4}, \frac{1}{8}, \frac{5}{8} \right)}$$

or approximately,

$$\boxed{\pi = (0.25, 0.125, 0.625)}.$$

5. Long-run distribution

The chain is irreducible and aperiodic. Therefore, the transition probabilities converge to the unique stationary distribution:

$$\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} \pi \\ \pi \\ \pi \end{pmatrix}.$$

The given value

$$P^{100} = \begin{pmatrix} 0.250 & 0.125 & 0.625 \\ 0.250 & 0.125 & 0.625 \\ 0.250 & 0.125 & 0.625 \end{pmatrix}$$

is already very close to this limiting matrix.

Therefore, the long-run distribution of the population is

$$\pi = (0.250, 0.125, 0.625).$$

Hence, in the long run,

$$25\% \text{ of the population is Susceptible,}$$

$$12.5\% \text{ is Sick,}$$

and

$$62.5\% \text{ is Immune.}$$

This means that, regardless of the initial distribution, the proportions of the population converge to these values as the number of days becomes large.

The population consists of 50 000 000 individuals.

Susceptible Individuals

$$\begin{aligned} N_S &= 50\,000\,000 \times 0.25 \\ &= 12\,500\,000. \end{aligned}$$

Sick Individuals

$$\begin{aligned} N_D &= 50\,000\,000 \times 0.125 \\ &= 6\,250\,000. \end{aligned}$$

Immune Individuals

$$\begin{aligned} N_I &= 50\,000\,000 \times 0.625 \\ &= 31\,250\,000. \end{aligned}$$

In the long run, the model predicts approximately 12.5 million Susceptible individuals, 6.25 million Sick individuals, and 31.25 million Immune individuals.