

Analysis I

Solutions of Tutorial Exercise Sheet 1

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This document is supplemented for the first chapter lecture notes (Analysis 1).

Solution of exercise 01:

- (1) $S_1 = \{x \in \mathbb{R} : x < 2\} =] - \infty, 2[$
- (2) $S_2 = \{x \in \mathbb{R} : -x \leq 2\} = [-2, +\infty[$
- (3) $S_3 = \{x \in \mathbb{R} : x^2 \geq 3\} =] - \infty, -\sqrt{3}] \cup [\sqrt{3}, +\infty[$
- (4) $S_4 = \{x \in \mathbb{R} : x^2 \leq 2\} = [-\sqrt{2}, \sqrt{2}]$
- (5) $S_5 = \{x \in \mathbb{R} : x^2 \geq 2\} =] - \infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty[$
- (6) $S_6 = \{x \in \mathbb{R} : -x^3 \geq 3\} =] - \infty, -\sqrt[3]{3}]$

Solution of exercise 02:

A)

1. For all $x, y \in \mathbb{R}$:

$$\begin{aligned} 2|x| &= |(x+y) + (x-y)| \leq |x+y| + |x-y| \\ 2|y| &= |(x+y) + (y-x)| \leq |x+y| + |x-y| \end{aligned}$$

Adding these two inequalities:

$$|x| + |y| \leq |x+y| + |x-y|, \quad \forall x, y \in \mathbb{R}.$$

2. $\forall x, y \geq 0$, since $2\sqrt{xy} \geq 0$:

$$x + y \leq x + 2\sqrt{xy} + y = (\sqrt{x} + \sqrt{y})^2$$

Taking square roots: $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.

3. For all $x, y \geq 0$: since $x = (x - y) + y \leq |x - y| + y$, applying the result of (2):

$$\sqrt{x} \leq \sqrt{|x - y|} + \sqrt{y} \implies \sqrt{x} - \sqrt{y} \leq \sqrt{|x - y|} \quad (*)$$

Similarly:

$$\sqrt{y} - \sqrt{x} \leq \sqrt{|x - y|} \quad (**)$$

Combining (*) and (**):

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}.$$

B)

• **Values:**

$$\lfloor 3.6 \rfloor = 3, \quad \lfloor \pi \rfloor = 3, \quad \lfloor e \rfloor = 2, \quad \lfloor -5.3 \rfloor = -6, \quad \lfloor -0.4 \rfloor = -1, \quad \lfloor 8 \rfloor = 8.$$

• **Proofs:**

1. Since $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, we have:

$$\lfloor x \rfloor + m \leq x + m < \lfloor x \rfloor + m + 1$$

Since $\lfloor x + m \rfloor$ is the largest integer $\leq x + m$, we get $\lfloor x \rfloor + m \leq \lfloor x + m \rfloor$.

Since $\lfloor x + m \rfloor + 1$ is the smallest integer $> x + m$, we get $\lfloor x + m \rfloor \leq \lfloor x \rfloor + m$.

Therefore: $\lfloor x + m \rfloor = \lfloor x \rfloor + m$.

2. If $x \leq y$, then:

$$\lfloor x \rfloor \leq x \leq y < \lfloor y \rfloor + 1$$

Since $\lfloor x \rfloor$ is an integer and $\lfloor y \rfloor$ is the greatest integer $\leq y$:

$$\lfloor x \rfloor \leq \lfloor y \rfloor.$$

3. From $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ and $\lfloor y \rfloor \leq y < \lfloor y \rfloor + 1$:

$$\lfloor x \rfloor + \lfloor y \rfloor \leq x + y < \lfloor x \rfloor + \lfloor y \rfloor + 2$$

Since $\lfloor x + y \rfloor$ is the greatest integer $\leq x + y$:

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \quad \text{and} \quad \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1.$$

Solution of exercise 03:

Let A and B be two non-empty subsets of $\mathbb{R}$ such that $B \subset A$.

1. Show that: A is bounded $\implies B$ is bounded

Definition: A subset $E \subset \mathbb{R}$ is **bounded** if it is both bounded above and bounded below. That is, there exist two real numbers $m, M \in \mathbb{R}$ such that:

$$\forall x \in E, \quad m \leq x \leq M$$

Suppose A is bounded. By definition, there exist two real numbers $m, M \in \mathbb{R}$ such that:

$$\forall x \in A, \quad m \leq x \leq M$$

Let $y \in B$. Since $B \subset A$, it follows that $y \in A$. Therefore, the inequality applies to y :

$$m \leq y \leq M$$

Since this holds for every $y \in B$, the subset B is bounded below by m and bounded above by M . Hence, B is bounded.

2. Show that $\inf(A) \leq \inf(B)$ and $\sup(A) \geq \sup(B)$

(a) Proof for the infimum: $\inf(A) \leq \inf(B)$

Let $m_A = \inf(A)$. By definition, m_A is a lower bound of A , meaning:

$$\forall x \in A, \quad m_A \leq x$$

Since $B \subset A$, for all $y \in B$, we have $y \in A$, which implies:

$$\forall y \in B, \quad m_A \leq y$$

This shows that $m_A = \inf(A)$ is a lower bound of B .

By definition, $\inf(B)$ is the *greatest lower bound* of B . Consequently, any lower bound of B must be less than or equal to $\inf(B)$. Thus:

$$\inf(A) \leq \inf(B)$$

(b) Proof for the supremum: $\sup(A) \geq \sup(B)$

Let $M_A = \sup(A)$. By definition, M_A is an upper bound of A , meaning:

$$\forall x \in A, \quad x \leq M_A$$

Since $B \subset A$, for all $y \in B$, we have $y \in A$, which implies:

$$\forall y \in B, \quad y \leq M_A$$

This shows that $M_A = \sup(A)$ is an upper bound of B .

By definition, $\sup(B)$ is the *least upper bound* of B . Consequently, any upper bound of B must be greater than or equal to $\sup(B)$. Thus:

$$\sup(B) \leq \sup(A) \iff \sup(A) \geq \sup(B)$$

Solution of exercise 04:

1. Summary table:

A	Maj(A)	Min(A)	sup A	inf A	max A	min A
$[-\alpha, \alpha]$	$[\alpha, +\infty[$	$] - \infty, -\alpha]$	α	$-\alpha$	α	$-\alpha$
$[-\alpha, \alpha[$	$[\alpha, +\infty[$	$] - \infty, -\alpha]$	α	$-\alpha$	$\nexists$	$-\alpha$
$] - \alpha, \alpha]$	$[\alpha, +\infty[$	$] - \infty, -\alpha]$	α	$-\alpha$	α	$\nexists$
$] - \alpha, \alpha[$	$[\alpha, +\infty[$	$] - \infty, -\alpha]$	α	$-\alpha$	$\nexists$	$\nexists$

2. $A =] - \sqrt{2}, \sqrt{2}[$ (4th case in the table above).

3. $A = \left\{ \frac{n-1}{n}, n \in \mathbb{N}^* \right\}$

- $\forall n \in \mathbb{N}^* : n \geq 1 \Rightarrow \frac{n-1}{n} \geq 0$ and $0 \in A$ (for $n = 1$), hence $\min A = \inf A = 0$.
- $\sup A = 1$: We verify:

(a) $\forall n \in \mathbb{N}^* : \frac{n-1}{n} = 1 - \frac{1}{n} < 1$ so 1 is an upper bound.

(b) $\forall \varepsilon > 0$, choose $n_\varepsilon = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$, then $\frac{n_\varepsilon - 1}{n_\varepsilon} > 1 - \varepsilon$.

Since $1 \notin A$, $\max A$ does not exist.

Solution of exercise 05:

Let $M = \sup A$ and $m = \inf A$.

1. For all $(x, y) \in A^2$:

$$\begin{aligned}
 m &\leq x \leq M \quad \text{and} \quad m \leq y \leq M \\
 \Rightarrow -M &\leq -y \leq -m \Rightarrow -(M - m) \leq x - y \leq M - m \\
 &\Rightarrow |x - y| \leq M - m.
 \end{aligned}$$

Therefore $M - m$ is an upper bound for B .

2. We show $\sup B = M - m$:

- **Upper bound:** Shown above.

- **Characterization:** If $\sup A = M$, then $\forall \varepsilon > 0, \exists x \in A : M - \frac{\varepsilon}{2} < x$. If $\inf A = m$, then $\forall \varepsilon > 0, \exists y \in A : y < m + \frac{\varepsilon}{2}$.

Combining:

$$(M - m) - \varepsilon < x - y \leq |x - y|$$

Therefore $\sup B = M - m = \sup A - \inf A$.

Solution of exercise 06:

(1a) Since $A \subset A \cup B$ and $B \subset A \cup B$:

$$\sup A \leq \sup(A \cup B) \quad \text{and} \quad \sup B \leq \sup(A \cup B)$$

$$\Rightarrow \max(\sup A, \sup B) \leq \sup(A \cup B) \quad (*)$$

Conversely, $\forall x \in A \cup B$: $x \leq \sup A$ or $x \leq \sup B$, so $x \leq \max(\sup A, \sup B)$.

$$\Rightarrow \sup(A \cup B) \leq \max(\sup A, \sup B) \quad (**)$$

From (*) and (**): $\sup(A \cup B) = \max(\sup A, \sup B)$.

(1b) Similarly: $\inf(A \cup B) = \min(\inf A, \inf B)$.

(2a) Since $A \cap B \subset A$ and $A \cap B \subset B$:

$$\sup(A \cap B) \leq \sup A \quad \text{and} \quad \sup(A \cap B) \leq \sup B$$

$$\Rightarrow \sup(A \cap B) \leq \min(\sup A, \sup B).$$

(2b) Similarly: $\inf(A \cap B) \geq \max(\inf A, \inf B)$.

(3a) Let $M_A = \sup A$ and $M_B = \sup B$.

- $\forall x \in A : x \leq M_A$ and $\forall y \in B : y \leq M_B$

$$\Rightarrow \forall z \in A + B : z = x + y \leq M_A + M_B$$

- $\forall \varepsilon > 0, \exists x \in A : M_A - \frac{\varepsilon}{2} < x$ and $\exists y \in B : M_B - \frac{\varepsilon}{2} < y$

$$\Rightarrow \exists z = x + y \in A + B : (M_A + M_B) - \varepsilon < z$$

Therefore $\sup(A + B) = \sup A + \sup B$.

(3b) Similarly: $\inf(A + B) = \inf A + \inf B$.

(4a)

- $\forall x \in A : x \geq \inf A \Rightarrow -x \leq -\inf A$, so $-\inf A$ is an upper bound for $-A$, hence $\sup(-A) \leq -\inf A$.
- $\forall (-x) \in (-A) : -x \leq \sup(-A) \Rightarrow x \geq -\sup(-A)$, so $-\sup(-A)$ is a lower bound for A , hence $\inf A \geq -\sup(-A)$, i.e. $\sup(-A) \geq -\inf A$.

Therefore $\sup(-A) = -\inf A$.

(4b) Similarly: $\inf(-A) = -\sup A$.

Solution of exercise 07:

1. $A = \left\{ \frac{3n+1}{2n+1}, n \in \mathbb{N} \right\}$

- $\inf A = 1$: $\forall n \in \mathbb{N} : 3n \geq 2n \Rightarrow 3n+1 \geq 2n+1 \Rightarrow \frac{3n+1}{2n+1} \geq 1$. Since $1 \in A$ (for $n = 0$): $\min A = \inf A = 1$.

- $\sup A = \frac{3}{2}$: $\forall n \in \mathbb{N} : \frac{3n+1}{2n+1} < \frac{3}{2}$ (since $2(3n+1) = 6n+2 < 6n+3 = 3(2n+1)$).

For the characterization: $\forall \varepsilon > 0$, choose $n_\varepsilon = \left\lfloor \frac{|1-2\varepsilon|}{4\varepsilon} \right\rfloor + 1$. Then $\frac{3}{2} - \varepsilon < \frac{3n_\varepsilon+1}{2n_\varepsilon+1}$.

Since $\frac{3}{2} \notin A$: $\max A$ does not exist.

2. $B = \left\{ \frac{1}{n} + \frac{1}{n^2}, n \in \mathbb{N}^* \right\}$

- $\sup B = 2$: $\forall n \in \mathbb{N}^* : n \geq 1 \Rightarrow \frac{1}{n} \leq 1$ and $\frac{1}{n^2} \leq 1$, so $\frac{1}{n} + \frac{1}{n^2} \leq 2$. Since $2 \in B$ (for $n = 1$): $\max B = \sup B = 2$.

- $\inf B = 0$: $\forall n \in \mathbb{N}^* : \frac{1}{n} + \frac{1}{n^2} > 0$. For the characterization: choose $n_\varepsilon = \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1$.

Since $0 \notin B$: $\min B$ does not exist.

3. $C = \{e^{-n}, n \in \mathbb{N}\}$

- $\sup C = 1$: $\forall n \in \mathbb{N} : n \geq 0 \Rightarrow e^{-n} \leq e^0 = 1$. Since $1 \in C$ (for $n = 0$): $\max C = \sup C = 1$.

- $\inf C = 0$: $\forall n \in \mathbb{N} : e^{-n} > 0$. For the characterization: choose $n_\varepsilon = \lfloor \ln(\varepsilon) \rfloor + 1$. Since $0 \notin C$: $\min C$ does not exist.

4. $D = \left\{ \frac{1}{n^2} - 2, n \in \mathbb{N}^* \right\}$

- $\sup D = -1$: $\forall n \in \mathbb{N}^* : n \geq 1 \Rightarrow n^2 \geq 1 \Rightarrow \frac{1}{n^2} \leq 1 \Rightarrow \frac{1}{n^2} - 2 \leq -1$. Since $-1 \in D$ (for $n = 1$): $\max D = \sup D = -1$.

- $\inf D = -2$: $\forall n \in \mathbb{N}^* : \frac{1}{n^2} > 0 \Rightarrow \frac{1}{n^2} - 2 > -2$. For the characterization: choose $n_\varepsilon = \left\lfloor \sqrt{\frac{1}{\varepsilon}} \right\rfloor + 1$. Since $-2 \notin D$: $\min D$ does not exist.