

Solutions to tutorial exercises set(0) : (Analysis 1)

Exercise 01:

A. All these numbers are real, so they are $\mathbb{R}$:

(i) -2 is a negative natural number, i.e. an integer (a, b, c, d)

(ii) $\frac{1}{3}$ is a ratio of integers so it is a rational (c, d)

(iii) 0 is an integer (a, c, d) .

(iv) 7 is a natural number and an integer. It is in fact also a prime number - That is, only divisible by itself or 1 . It is also an odd number (can not be exactly divided by 2)

(a, c, d, g)

(v) $\frac{21}{5}$ is a rational number (actually an improper fraction)
 (c, d) .

numerator (21) , denominator (5)

(vi) $-\frac{3}{4}$ is a rational number (a proper fraction) (b, c, d)

numerator (-3) , denominator (4)

(vii) 0.73 is actually a decimal representation of a rational number ($0.73 = \frac{73}{100}$) sometimes called a decimal fraction, or simply a decimal; (c, d, f) .

(viii) 11 is a natural number and an integer (like 7 it is also prime, and is also odd as any prime greater than 2 must be,

(a, c, d, g)

(ix) 8 is another natural number and an integer (but it is not prime) since it can be written as $2 \times 2 \times 2 = 2^3$. It is also an even number (a, c, d) .

(x) The square root of 2 is not a rational number. (d, e)

(xi) -0.49 is a decimal representation of the negative rational number $(-\frac{49}{100})$ (b, c, d, f)

(xii) π , the ratio of the circumference of a circle to its diameter, is not a rational number (it is an irrational number). That is, it cannot be written as a fraction. $\frac{22}{7}$ for example, is just an approximation to π ; (d, e).

B. (i) $0 \times 1 = 0$, i.e. zero - which is also finite

(ii) $0 + 1 = 1$, finite, non-zero

(iii) $\frac{1}{0}$ does not exist - it is not infinite, negative, zero, finite or non-zero (b).

(iv) $2 - 0 = 2$, finite and non-zero (e, f).

(v) $0^2 = 0 \times 0 = 0$, zero and finite (d, e).

(vi) $0 - 1 = -1$, negative, finite, non-zero (c, e, f).

(vii) $\frac{0}{0}$ does not exist (you can't "cancel" the zeros!). It is not infinite, negative, zero, finite or non-zero - it just does not exist (b).

(viii) $\frac{3}{0} \times 0 + \frac{3}{0}$ does not exist because $\frac{3}{0}$ does not exist (b).

(ix) $\frac{0^3}{0}$ again, does not exist (b)

(x) $\frac{2}{2} = 1$ - no problem here, finite and non-zero (e, f)

Exercise 02%

1) $x > 0$

2) $1 < x < 0$

3) $-1 < x < 3$

4) $-2 \leq x < 2$

5) If the absolute value of x is less than 2 then this means that if x is positive then $0 \leq x < 2$, but if x is negative then we must have $-2 < x \leq 0$. So, combining these we must have $-2 < x < 2$. This can also be expressed in terms of modulus $|x| < 2$.

Exercise 03:

$$\begin{aligned} \text{(i)} \quad \frac{2}{x+1} - \frac{3}{x-2} &= \frac{2(x-2) - 3(x+1)}{(x+1)(x-2)} \\ &= \frac{2x - 4 - 3x - 3}{x^2 - 2x + x - 2} \\ &= \frac{-x - 7}{x^2 - x - 2} \\ &= -\frac{x+7}{x^2 - x - 2} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \frac{1}{x-1} + \frac{1}{x+1} - \frac{1}{x+2} &= \frac{(x+1) + (x-1)}{(x-1)(x+1)} - \frac{1}{(x+2)} \\ &= \frac{x+1+x-1}{(x^2-1)} - \frac{1}{(x+2)} \\ &= \frac{2x(x+2) - (x^2-1)}{(x^2-1)(x+2)} \\ &= \frac{2x^2 + 4x - x^2 + 1}{x^3 + 2x^2 - x - 2} \\ &= \frac{x^2 + 4x + 1}{x^3 + 2x^2 - x - 2} \end{aligned}$$

Exercise 04:

odd function: $f(-x) = -f(x)$

even function: $f(-x) = f(x)$

$$\text{(i)} \quad 3x^3 - x = f(x) \Rightarrow f(-x) = 3(-x)^3 - (-x) = -(3x^3 - x) = -f(x)$$

$3x^3 - x$ is an odd function

$$\text{(ii)} \quad \frac{x^2}{1+x^2} = f(x) \Rightarrow f(-x) = \frac{(-x)^2}{1+(-x)^2} = f(x)$$

$\frac{x^2}{1+x^2}$ is an even function

$$\text{(iii)} \quad \frac{2x}{x^2-1} = f(x) \Rightarrow f(-x) = \frac{2(-x)}{(-x)^2-1} = -\frac{2x}{x^2-1} = -f(x)$$

$$(iv) \frac{x^2}{x-1} = f(x) \Rightarrow f(-x) = \frac{(-x)^2}{(-x)-1} = \frac{x^2}{-x-1}$$

$\frac{x^2}{x-1}$ neither an even function nor an odd function.

Exercise 05:

(a) Let us suppose $\frac{a}{b}$ as a number (if it exists) such that $b \cdot x = a$, then $\frac{0}{0} = x$; where $0 \cdot x = 0$ but this is true for all numbers.
 $\Rightarrow$ it is undefined.

(b) Similar to (a), if we define $\frac{1}{0}$ as a number (if it exists) such that $0 \cdot x = 1$, we reach the conclusion that there is no satisfied this condition.

Exercise 06:

An odd integer: $n = 2k + 1$ where $k \in \mathbb{Z}$

We have: $(2k + 1)^2 = 4k^2 + 4k + 1$. Notice that, we can write:

$$4k^2 + 4k = 2(2k^2 + 2k) = 2p \text{ where } p = 2k^2 + 2k$$

So: $(2k + 1)^2 = 2p + 1$, which is an odd number.

Exercise 07:

We assume by contradiction that $\sqrt{2} \in \mathbb{Q}$ then:

$$\exists p, q \in \mathbb{Z}, \text{gcd}(p, q) = 1 \text{ such that } \sqrt{2} = \frac{p}{q} \Leftrightarrow p^2 = 2q^2$$

greatest common divisor

$\Rightarrow 2$ divides p since the square of an integer is even only if the integer itself is even

$$\text{so } p = 2k \text{ where } k \in \mathbb{Z} \Rightarrow 4k^2 = 2q^2 \Rightarrow 2k^2 = q^2 \Rightarrow 2 \text{ divides } q^2 \Rightarrow$$

2 divides q , which is a contradiction since p and q are relatively prime.

Exercise 08:

Since $\frac{p}{q}$ is a root, on substituting in the given equation, we find:

$$a_0 \left(\frac{p}{q}\right)^n + a_1 \left(\frac{p}{q}\right)^{n-1} + a_2 \left(\frac{p}{q}\right)^{n-2} + \dots + a_{n-1} \left(\frac{p}{q}\right) + a_n = 0$$

-4-

multiplying by q^n : $a_0 p^n + a_1 p^{n-1} q + a_2 p^{n-2} q^2 + \dots + a_{n-1} p q^{n-1} + a_n q^n = 0$ ---(*)

dividing by p : $a_0 p^{n-1} + a_1 p^{n-2} q + \dots + a_{n-1} q^{n-1} = -\frac{a_n q^n}{p}$

Notice that, $a_0 p^{n-1} + a_1 p^{n-2} q + \dots + a_{n-1} q^{n-1}$ is an integer

$\Rightarrow -\frac{a_n q^n}{p}$ is also integer. Then since p and q are relatively prime
 $\Rightarrow p$ does not divide q^n exactly and
 $\Rightarrow$ must divide a_n

In the other hand, by transposing the first term of (*):

$$a_1 p^{n-1} q + a_2 p^{n-2} q^2 + \dots + a_{n-1} p q^{n-1} + a_n q^n = -a_0 p^n$$

and dividing by q :

$$a_1 p^{n-1} + a_2 p^{n-2} q + \dots + a_{n-1} p q^{n-2} + a_n q^{n-1} = -\frac{a_0 p^n}{q}$$

We find that: $a_1 p^{n-1} + a_2 p^{n-2} q + \dots + a_{n-1} p q^{n-2} + a_n q^{n-1} \in \mathbb{Z}$

so $-\frac{a_0 p^n}{q} \in \mathbb{Z}$

$\Rightarrow q$ must divide a_0 (since p and q are relatively prime so
 q does not divide p^n exactly and
 $\Rightarrow$ must divide a_0)

Exercise 09:

Let us take: $x = \sqrt{2} + \sqrt{3} \Rightarrow x^2 = (\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$

$\Leftrightarrow x^2 - 5 = 2\sqrt{6}$ and squaring $(x^2 - 5)^2 = x^4 - 10x^2 + 25 = 4 \times 6 = 24$

$\Rightarrow x^4 - 10x^2 + 1 = 0$

We use the result in exercise 08. Notice that, $a_0 = 1$ and $a_n = 1$

so if $\frac{p}{q}$ is a root of $x^4 - 10x^2 + 1 = 0$, we have:

p must divide 1 $\Rightarrow p = \pm 1$

q must divide 1 $\Rightarrow q = \pm 1$

Then, the only possible rational roots of the equation

$x^4 - 10x^2 + 1 = 0$ are ± 1 and these do not satisfy the equation

$\Rightarrow \sqrt{2} + \sqrt{3}$ cannot be rational number.

Exercise 10:

Let us assume $a < b$ adding a to both side: $2a < a+b \Leftrightarrow a < \frac{a+b}{2}$

In the other hand, adding b to both sides $a+b < 2b$ and $\frac{a+b}{2} < b$

Thus $a < \frac{a+b}{2} < b$.

Let us now, prove that $\frac{a+b}{2} \in \mathbb{Q}$: Use take: $\begin{cases} a = \frac{p}{q} \\ b = \frac{r}{s} \end{cases}$

where $p, q, r, s \in \mathbb{Z}$

$$\Rightarrow \frac{a+b}{2} = \frac{1}{2} \left(\frac{p}{q} + \frac{r}{s} \right) = \frac{1}{2} \left(\frac{ps+qr}{qs} \right) = \frac{ps+qr}{2qs}$$

where $\begin{cases} (ps+qr) \in \mathbb{Z} \\ \text{and} \\ 2qs \in \mathbb{Z} \end{cases}$

$$\Rightarrow \frac{ps+qr}{2qs} \in \mathbb{Q}$$

Exercise 06

1) - using proof by contradiction.

We assume that $x + v \in \mathcal{O}$

We have $v \in \mathcal{O}$, so $\exists p, q \in \mathbb{Z}$

such that $v = \frac{p}{q}$, $q \neq 0$

and $x + v \in \mathcal{O} \Rightarrow \exists p', q' \in \mathbb{Z}$, $x + v = \frac{p'}{q'}$

such that $q' \neq 0$

which implies $x + \frac{p}{q} = \frac{p'}{q'} \Rightarrow x = \frac{p}{q} - \frac{p'}{q'}$
 $\Rightarrow x = \frac{pq' - p'q}{qq'}$

and $q \cdot q' \neq 0$.

$\Rightarrow x \in \mathcal{O}$; this is contradiction
with $x \notin \mathcal{O}$.

2) prove that $\sqrt{2} \notin \mathcal{O}$

by contradiction, we assume

that $\sqrt{2} \in \mathcal{O}$.

then $\exists p, q \in \mathbb{Z}$, $q \neq 0$, $\text{GCD}(p, q) = 1$

such that $\sqrt{2} = \frac{p}{q}$. (p, q are
prime)

so

$$\sqrt{2} = \frac{p}{q} \Rightarrow \sqrt{2} \cdot q = p$$

$$\Rightarrow 2q^2 = p^2$$

$$\Rightarrow p^2 \text{ is even}$$

$$\Rightarrow p \text{ is even}$$

$$\Rightarrow p = 2 \cdot k, k \in \mathbb{Z}$$

$$\text{Thus } 2q^2 = (2k)^2 \Rightarrow q^2 = 2k^2$$

$$\Rightarrow q^2 = 2 \cdot k^2, k \in \mathbb{Z}$$

$$\Rightarrow q^2 \text{ is even}$$

$$\Rightarrow q \text{ is even}$$

this contradiction because
 p and q are prime then $\sqrt{2} \notin \mathcal{O}$

$$3) \sqrt{2} + \sqrt{3} \notin \mathcal{O}$$

let us take

$$x = \sqrt{2} + \sqrt{3} \Rightarrow x^2 = (\sqrt{2} + \sqrt{3})^2$$

$$\Rightarrow x^2 = 2\sqrt{2}\sqrt{3} + 2 + 3$$

$$\Rightarrow x^2 = 5 + 2\sqrt{6}$$

$$\Rightarrow x^2 - 5 = 2\sqrt{6}$$

$$\Rightarrow (x^2 - 5)^2 = (2\sqrt{6})^2$$

$$\Rightarrow x^4 - 10x^2 + 25 = 24$$

$$\Rightarrow x^4 - 10x^2 + 1 = 0$$

We use the results in exercise 5,

Notice that:

$$a_0 = 1, a_n = 1$$

So if $x = \frac{p}{q}$ is root of

$$x^4 - 10x^2 + 1 = 0, \text{ we have}$$

p must divide 1 $\Rightarrow p = \pm 1$

q must divide 1 $\Rightarrow q = \pm 1$

then, the only possible rational roots of the equation

$$x^4 - 10x^2 + 1 = 0 \text{ are } \pm 1$$

these not satisfy the equation

then $\sqrt{2} + \sqrt{3}$ can not be rational number.