

Chapter 1: Field of Real Numbers

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1 Field of real numbers

1.1 Definition and usual properties of the field of real numbers

Definition 1.1. *The field of real numbers $(\mathbb{R}, +, \times, \leq)$ is the set of real numbers under the two operations of addition and multiplication, with an ordered ring structure of $\mathbb{R}$.*

The field of real numbers $\mathbb{R}$ has the following usual properties:

1. $\mathbb{R}$ is a commutative field; i.e., addition and multiplication are both compatible with the order structure:
 - 1.1. Addition and multiplication are both commutative, which means that $x + y = y + x$ and $x \times y = y \times x$, for every real numbers x and y .
 - 1.2. Addition and multiplication are both associative, which means that $(x + y) + z = x + (y + z)$ and $(x \times y) \times z = x \times (y \times z)$, for every real numbers x , y and z .
 - 1.3. There is a real number called zero and denoted 0 which is an additive identity, which means that $x + 0 = 0 + x = x$, for every real number x .
 - 1.4. There is a real number denoted 1 which is a multiplicative identity, which means that $1 \times x = x \times 1 = x$, for every real number x .
 - 1.5. Every real number x has an additive inverse denoted $-x$. This means that $x + (-x) = 0$ for every real number x .
 - 1.6. Every nonzero real number x has a multiplicative inverse denoted x^{-1} or $\frac{1}{x}$. This means that $x \times x^{-1} = 1$ for every nonzero real number x .
 - 1.7. Multiplication is distributive over addition, which means that $x \times (y + z) = x \times y + x \times z$, for every real numbers x , y and z .
2. The field $\mathbb{R}$ is ordered, meaning that there is a total order $\leq$ such that:
 - 2.1. Reflexive relationship; i.e., $x \leq x$ for every real number x .
 - 2.2. Antisymmetric relationship; $x \leq y \wedge y \leq x \implies x = y$, for every real numbers x and y .
 - 2.3. Transitive relationship; $x \leq y \wedge y \leq z \implies x \leq z$, for every real numbers x , y and z .
3. Many other properties can be deduced from the above ones. In particular:
 - 3.1. For all $x, y \in \mathbb{R}$, $x \geq 0 \wedge y \geq 0 \implies x \cdot y \geq 0$.
 - 3.2. For all $x, y \in \mathbb{R}$, $(x \leq y)$ or $(y \leq x)$.
 - 3.3. For all $x, y, z \in \mathbb{R}$, $x \leq y \implies x + z \leq y + z$.
 - 3.4. For all $n, m \in \mathbb{N}$ and for all $x \in \mathbb{R}_+$:

$$\begin{cases} x \leq 1 \text{ and } n \leq m \implies x^n \geq x^m, \\ x \geq 1 \text{ and } n \leq m \implies x^n \leq x^m \end{cases}$$

- 3.5. For all $x, y \in \mathbb{R}^*$:

$$\begin{cases} 0 < x \leq y \implies 0 < \frac{1}{y} \leq \frac{1}{x}, \\ x \leq y < 0 \implies \frac{1}{y} \leq \frac{1}{x} < 0, \\ x < 0 < y \implies \frac{1}{x} < 0 < \frac{1}{y} \end{cases}$$

1.2 Newton's binomial formula

Proposition 1.1. *Let x and y be two real numbers and n be a non-zero natural number. The Newton's binomial formula is given by*

$$(x + y)^n = \sum_{k=0}^n C_n^k x^k y^{n-k}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$, $1! = 1$ and $0! = 1$.

Notation 1.2.1 :

- $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$
- $\mathbb{R}_+^* = \{x \in \mathbb{R}; x > 0\}$
- $\mathbb{R}_-^* = \{x \in \mathbb{R}; x < 0\}$
- $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$
- $\mathbb{R}_- = \{x \in \mathbb{R}; x \leq 0\}$

1.3 Intervals of $\mathbb{R}$

Let $a, b \in \mathbb{R}$ be two real numbers, such that $a < b$. The intervals of $\mathbb{R}$ are:

1. $\mathbb{R} =]-\infty, +\infty[$.
2. The empty set.
3. $[a, b] = \{x \in \mathbb{R}; a \leq x \leq b\}$.
4. $]a, b[= \{x \in \mathbb{R}; a < x < b\}$.
5. $]a, b] = \{x \in \mathbb{R}; a < x \leq b\}$.
6. $[a, b[= \{x \in \mathbb{R}; a \leq x < b\}$.
7. $[a, +\infty[= \{x \in \mathbb{R}; x \geq a\}$.
8. $]a, +\infty[= \{x \in \mathbb{R}; x > a\}$.
9. $] -\infty, b] = \{x \in \mathbb{R}; x \leq b\}$.
10. $] -\infty, b[= \{x \in \mathbb{R}; x < b\}$.

1.4 Completed number line $\overline{\mathbb{R}}$ (Extension of $\mathbb{R}$)

Definition 1.2. *We call the completed number line $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{-\infty, +\infty\}$.*

1.5 Operations on $\overline{\mathbb{R}}$

1. $(+\infty) + (+\infty) = +\infty$ and $(-\infty) + (-\infty) = -\infty$.
2. For all $x \in \mathbb{R}$, we have $x + (+\infty) = +\infty$ and $x + (-\infty) = -\infty$.
3. $(+\infty) \times (+\infty) = +\infty$, $(-\infty) \times (-\infty) = +\infty$, $(+\infty) \times (-\infty) = -\infty$.
4. For all $x \in \mathbb{R}_-^*$, we have $x \times (+\infty) = -\infty$ and $x \times (-\infty) = +\infty$.
5. For all $x \in \mathbb{R}_+^*$, we have $x \times (+\infty) = +\infty$ and $x \times (-\infty) = -\infty$.

1.6 Indeterminate forms

It may often be possible to simply add, subtract, multiply, divide or exponentiate the corresponding limits of two functions. However, there are occasions where it is unclear what the sum, difference, product or power of these two limits ought to be. For example, it is unclear what the following expressions ought to evaluate to:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \times \infty, \quad (+\infty) + (-\infty)$$

1.7 Bounded subset, upper and lower bounds

1. A non-empty subset A ($A \subseteq \mathbb{R}$) is said to be bounded above if:

$$\exists M \in \mathbb{R}, \forall x \in A, x \leq M$$

In this case, the number M is called an upper bound of A ; i.e., $M \in \mathbb{R}$ is an upper bound of $A \iff \forall x \in A, x \leq M$.

2. A non-empty subset A ($A \subseteq \mathbb{R}$) is said to be bounded below if:

$$\exists m \in \mathbb{R}, \forall x \in A, x \geq m$$

In this case, the number m is called a lower bound of A ; i.e., $m \in \mathbb{R}$ is a lower bound of $A \iff \forall x \in A, x \geq m$.

3. A non-empty subset A ($A \subseteq \mathbb{R}$) is said to be bounded if and only if it is bounded above and below; i.e., $A \subseteq \mathbb{R}$ is bounded $\iff \exists m, M \in \mathbb{R}, \forall x \in A, m \leq x \leq M$.
4. A non-empty subset A ($A \subseteq \mathbb{R}$) that is not bounded is said to be unbounded.

1.7.1 Supremum and infimum

1. Let $A \subseteq \mathbb{R}$ be bounded above. The supremum of A (abbreviated $\sup A$) is the least upper bound of A that is greater than or equal to each element of A (i.e., $\forall x \in A, \sup A \geq x$).
2. Let $A \subseteq \mathbb{R}$ be bounded below. The infimum of A (abbreviated $\inf A$) is the greatest lower bound of A that is less than or equal to each element of A (i.e., $\forall x \in A, \inf A \leq x$).
3. Any nonempty subset A of $\mathbb{R}$ and bounded above admits a least upper bound ($\sup A$).
4. Any nonempty subset A of $\mathbb{R}$ and bounded below admits a greatest lower bound ($\inf A$).
5. When the supremum (respectively, infimum) exists, it is unique.
6. The supremum $\sup A$ (respectively, $\inf A$) does not necessarily belong to A .

1.7.2 Properties of supremum and infimum

Let A and B be two non-empty subsets of $\mathbb{R}$ that are bounded.

1. **Properties for Union ($A \cup B$):**

- If $A \cup B$ is bounded above, then:

$$\sup(A \cup B) = \max(\sup A, \sup B)$$

- If $A \cup B$ is bounded below, then:

$$\inf(A \cup B) = \min(\inf A, \inf B)$$

2. Properties for Intersection ($A \cap B$):

- If $A \cap B$ is non-empty and bounded above, then:

$$\sup(A \cap B) \leq \min(\sup A, \sup B)$$

- If $A \cap B$ is non-empty and bounded below, then:

$$\inf(A \cap B) \geq \max(\inf A, \inf B)$$

3. Monotonicity (Inclusion):

- If $A \subseteq B$, then $\sup A \leq \sup B$ (assuming B is bounded above) and $\inf A \geq \inf B$ (assuming B is bounded below).

Remark 1.1. Denote $A = \{x \in \mathbb{R}; x \in A\}$ and $-A = \{x \in \mathbb{R}; -x \in A\}$. We have:

$$\sup(A) = M \iff \inf(-A) = -M \iff \inf(-A) = -\sup(A) \iff \sup(-A) = -\inf(A)$$

1. The upper bounds of $A =]-2, 8]$ is $[8, +\infty[$, thus $\sup A = 8$.
2. The lower bounds of $A =]-2, 8]$ is $[-\infty, -2]$, thus $\inf A = -2$.
3. The upper bounds of $A =]-\infty, 3]$ is $[3, +\infty[$, thus $\sup A = 3$ and $\inf A$ does not exist.
4. The lower bounds of $A =]1, +\infty[$ is $]-\infty, 1]$, thus $\inf A = 1$ and $\sup A$ does not exist.

1.7.3 Upper bound and lower bound characteristic properties

Theorem 1.1. Let $A \subset \mathbb{R}$ be non-empty and let $M \in \mathbb{R}$.

$$M = \sup A \iff \begin{cases} 1) \forall x \in A, x \leq M, \\ 2) \forall \epsilon > 0, \exists x \in A, M - \epsilon < x \end{cases}$$

Theorem 1.2. Let $A \subset \mathbb{R}$ be non-empty and let $m \in \mathbb{R}$.

$$m = \inf A \iff \begin{cases} 1) \forall x \in A, x \geq m, \\ 2) \forall \epsilon > 0, \exists x \in A, m + \epsilon > x \end{cases}$$

1.7.4 Maximal and minimal elements

Definition 1.3. 1. Let $A \subset \mathbb{R}$ be non-empty and let $M \in \mathbb{R}$ be an upper bound of A , if $M \in A$ then we say that M is maximal of A and we note $\max A$.

2. Let $A \subset \mathbb{R}$ be non-empty and let $m \in \mathbb{R}$ be a lower bound of A , if $m \in A$ then we say that m is minimal of A and we note $\min A$.

Remark 1.2. • If $\max A$ exist, then $\sup A$ exist and $\max A = \sup A$.

- If $\min A$ exist, then $\inf A$ exist and $\min A = \inf A$.
- If $\sup A$ exist and $\sup A \notin A$ then $\max A$ does not exist.
- If $\inf A$ exist and $\inf A \notin A$ then $\min A$ does not exist.

Exemple 1.1. 1. Let $A =]-\infty, 3]$. We have $\sup A = 3$ and $3 \in A$, then $\max A = 3$.

2. Let $A = [-1, +\infty[$. We have $\inf A = -1$ and $-1 \in A$, then $\min A = -1$.

3. $A =]-2, 5[$. We have $\sup A = 5 \notin A$ and $\inf A = -2 \notin A$ then $\min A$ and $\max A$ do not exist.

1.8 Archimedes axiom

For all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$; (i.e., the set $\mathbb{N}$ is not bounded from above in $\mathbb{R}$).

1.9 Rational and irrational numbers

Definition 1.4. We note by $\mathbb{Q} = \{\frac{a}{b}; a \in \mathbb{Z}, b \in \mathbb{Z}^*\}$ and $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. The elements of $\mathbb{Q}$ are called rational numbers.

- The set which contains $\mathbb{Z}$, is stable for the $+$ and $\times$ laws.
- Provided with the restrictions of these laws; it is itself a commutative field.
- In particular the inverse of any element of $\mathbb{Q}^*$ is still in $\mathbb{Q}^*$.

Definition 1.5. The elements of $\mathbb{R} \setminus \mathbb{Q}$ are called irrational numbers.

1.9.1 Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 1.3. Let x, y be two real numbers, such that $x < y$, there exists a rational number $q \in \mathbb{Q}$, such that $x < q < y$; i.e., between two real numbers there is always a rational number. We translate this property by saying that $\mathbb{Q}$ is dense in $\mathbb{R}$.

1.10 Absolute value

Definition 1.6. We call absolute value of a real number x any application $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

which satisfies the following properties:

1. $|x| = 0 \iff x = 0$, for every real number x .
2. $|x + y| \leq |x| + |y|$, for every real numbers x and y .
3. $|x - y| \geq ||x| - |y||$, for every real numbers x and y .
4. $|x \times y| = |x| \times |y|$, for every real numbers x and y .

Proofs of absolute value properties

Let $x, y \in \mathbb{R}$. The properties of the absolute value are proven as follows:

1. If $x = 0$, then $|0| = 0$ by definition. Conversely, if $|x| = 0$, then if $x > 0$ we would have $|x| = x = 0$ (contradiction since $x > 0$), and if $x < 0$ we would have $|x| = -x = 0 \implies x = 0$. Thus, $x = 0$.
2. For any real number x , we have $-|x| \leq x \leq |x|$. Similarly, for y , $-|y| \leq y \leq |y|$. Adding these inequalities gives: $-(|x| + |y|) \leq x + y \leq |x| + |y|$. By definition of the absolute value, this implies that $|x + y| \leq |x| + |y|$.
3. Using the triangle inequality, we can write:
 $|x| = |x - y + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|$, and for y , we get $|y| - |x| \leq |y - x| = |x - y|$. Combining these two yields $-|x - y| \leq |x| - |y| \leq |x - y|$, which means that $||x| - |y|| \leq |x - y|$.

4. We know that for any real number z , $|z|^2 = z^2$. Therefore: $|x \times y|^2 = (x \times y)^2 = x^2 \times y^2 = |x|^2 \times |y|^2 = (|x| \times |y|)^2$. Since absolute values are non-negative, taking the square root yields $|x \times y| = |x| \times |y|$.

1.11 Integer part of a real number

For all $x \in \mathbb{R}$ there exists a unique integer ($n \in \mathbb{Z}$) noted $E(x)$ (or $[x]$) which is called integer part of x , satisfying:

$$\forall x \in \mathbb{R}, E(x) \leq x < E(x) + 1$$

In other words $E(x)$ is the largest integer less than or equal to x ($n \leq x < n + 1; n \in \mathbb{Z}$). Especially, if $x \in \mathbb{Z}$ then $E(x) = x$.

Properties

Let $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}$. The integer part $E(x)$ satisfies the following usual properties:

1. $x - 1 < E(x) \leq x$.
2. $E(x) = n \iff n \leq x < n + 1$.
3. $E(x + n) = E(x) + n$.
4. $E(x) + E(y) \leq E(x + y) \leq E(x) + E(y) + 1$.
5. If $x \leq y$, then $E(x) \leq E(y)$.

Example 1.2. 1. $E(0.3) = 0$ ($0 \leq 0.3 < 0 + 1 = 1$).

2. $E(3.3) = 3$ ($3 \leq 3.3 < 3 + 1 = 4$).

3. $E(-1.5) = -2$ ($-2 \leq -1.5 < -2 + 1 = -1$).

4. $E(-4) = -4$, $E(5) = 5$.

Exercise 1.1. Let $A = \left\{ x_n = \frac{1}{2} + \frac{n}{2n+1}; n \in \mathbb{N} \right\}$.

1. Prove that: $\forall x_n \in A, \frac{1}{2} \leq x_n < 1$.
2. Find $\inf(A)$ and $\sup(A)$.
3. Show that: $\sup(A) = 1$.

Solution:

1. We prove that $\frac{1}{2} \leq x_n < 1$ for all $n \in \mathbb{N}$. We have $\forall n \in \mathbb{N}, x_n = \frac{1}{2} + \frac{n}{2n+1}$. Since $\forall n \in \mathbb{N}, 0 \leq 2n < 2n + 1$, we get $0 \leq \frac{2n}{2n+1} < 1$. Dividing and rearranging gives $\frac{n}{2n+1} < \frac{1}{2}$ for all $n \in \mathbb{N}$. Adding $\frac{1}{2}$ to all parts yields $\frac{1}{2} \leq \frac{1}{2} + \frac{n}{2n+1} < 1$ for all $n \in \mathbb{N}$. Thus, $\frac{1}{2} \leq x_n < 1$ for all $n \in \mathbb{N}$.
2. Since $\frac{1}{2} \leq x_n < 1$, the set A is bounded; therefore, $\inf A$ and $\sup A$ exist. $x_0 = \frac{1}{2}$ is a lower bound of A and $x_0 = \frac{1}{2} \in A$, thus $\min A = \frac{1}{2}$, which means that $\inf A = \frac{1}{2}$. And 1 is the least upper bound of A , so $\sup A = 1$.

3. We will now show that $\sup A = 1$. To do this, we must check the second property of the supremum of the set A . Let $x_n = \frac{1}{2} + \frac{n}{2n+1} > 1 - \epsilon$, and we find n as a function of ϵ :

$$\begin{aligned}
 x_n = \frac{1}{2} + \frac{n}{2n+1} > 1 - \epsilon &\implies -\frac{1}{2} + \frac{n}{2n+1} > -\epsilon \\
 \implies \frac{1}{2} - \frac{n}{2n+1} < \epsilon &\implies \frac{2n+1-2n}{2(2n+1)} < \epsilon \\
 \implies \frac{1}{2(2n+1)} < \epsilon &\implies \frac{1}{2n+1} < 2\epsilon \\
 \implies 2n+1 > \frac{1}{2\epsilon} &\implies 2n > \frac{1}{2\epsilon} - 1 \implies n > \frac{1}{4\epsilon} - \frac{1}{2}
 \end{aligned}$$

Thus, $\exists n = E\left(\frac{1}{4\epsilon} - \frac{1}{2}\right) + 1$, such that $x_n > 1 - \epsilon$, which implies that $\sup A = 1$.