

Centre Universitaire Abdelhafid Boussouf — Mila

Institute of Science and Technology · Department of Civil and Hydraulic Engineering

Strength of Materials II

Chapter 1

In-Plane Bending of Symmetrical Beams

2 weeks · 3 hours of lecture · one worked example per method · bilingual speaker notes

Two sessions, one objective

Session 1

1 h 30

Internal forces

- Supports, the cohesion torsor, the notation $FR \leftrightarrow EN$
- The four simple loadings read off the torsor; the three kinds of bending
- Sign conventions, the differential relations, the area method
- Two reference cases, then Example A worked in full

Session 2

1 h 30

Stresses

- What happens inside the beam; Bernoulli, $\Delta\phi$ and the flexure formula
- Section modulus — where to put the material
- Jouravski's formula and shear flow
- Examples B and C, and an application: choosing an IPE section

By the end you must be able to: draw V and M_z for any statically determinate beam, size a section against $[\sigma]$, and check it against $[\tau]$ — in that order.

1

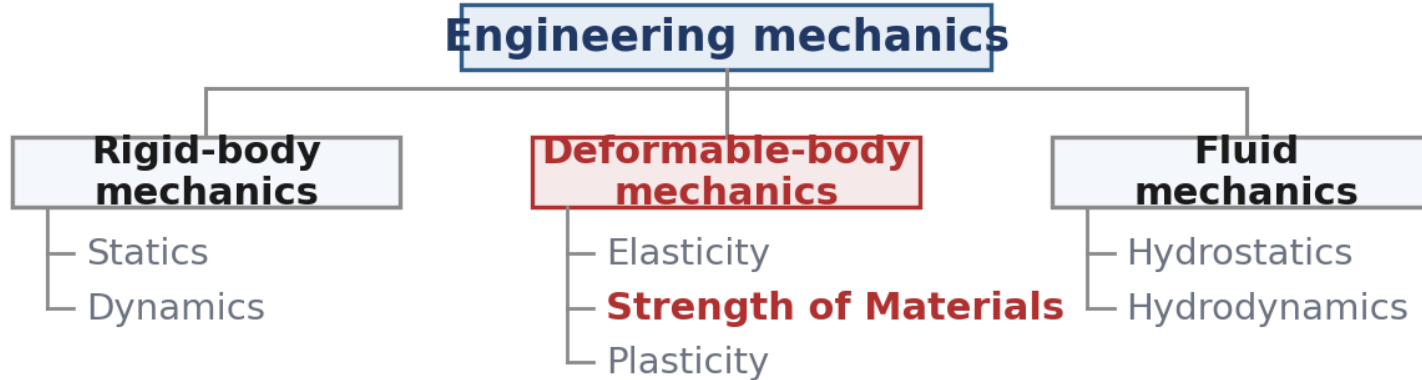
Reminders from Strength of Materials

1

Supports · internal forces · the four simple loadings

1 What Strength of Materials is

Session 1



Strength of Materials sets the shapes, the dimensions and the materials of a component, so that its strength and its deformation are both under control

It borrows from three places

Statics gives the external reactions. Kinematics of deformation gives the strains and displacements. The material law — E , G , ν , σ_e — links stress to strain. Strength of Materials puts the three together and turns them into dimensions.

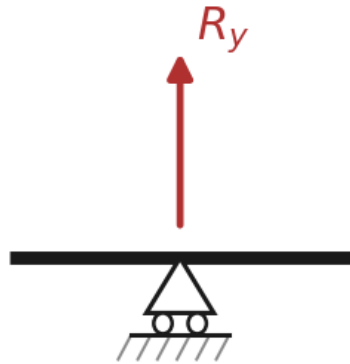
Two questions, always

- Is it strong enough? → compare the stress with an allowable stress
- Is it stiff enough? → compare the deflection with a limit such as $L/300$
- A member that passes the first and fails the second is still unacceptable.
- A floor beam that carries the load safely but bounces under footsteps has failed its user.

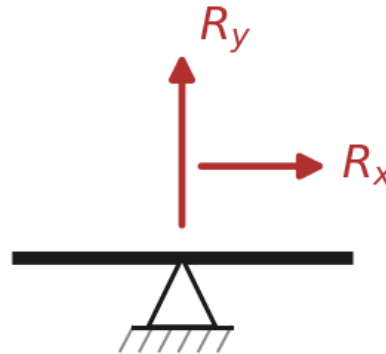
1 Supports and their reactions

Session 1

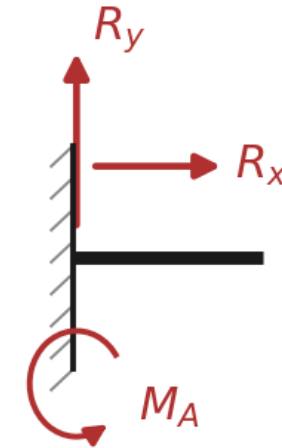
Roller
1 unknown



Pinned
2 unknowns



Fixed (built-in)
3 unknowns



Counting the unknowns

A plane structure has three equations of equilibrium: $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M = 0$.

Three reaction components $\rightarrow$ statically determinate (isostatique).

More than three $\rightarrow$ statically indeterminate (hyperstatique): equilibrium alone is not enough, and the compatibility of the deformations must be used — Chapter 5.

1 Two schools, one mechanics — the notation

Session 1

Quantity	En français	Used here (EN)	Your notes (FR)
Normal force	<i>effort normal</i>	N	N
Shear force	<i>effort tranchant</i>	V	T _y (or T)
Bending moment	<i>moment fléchissant</i>	M_z	M _{fz}
Torque	<i>moment de torsion</i>	M_t	M _t
Second moment of area	<i>moment quadratique</i>	I_z	I _{Gz}
First moment of area	<i>moment statique</i>	S*_z	W _{Gz}
Section modulus	<i>module de flexion</i>	W_z = I_z / y_{max}	I _{Gz} / y _{max}
Allowable stress	<i>contrainte pratique</i>	[σ] = σ_e / s	R _{pe} = R _e / s
Allowable shear stress	<i>contrainte pratique au cisaillement</i>	[τ]	R _{pg} = R _{eg} / s

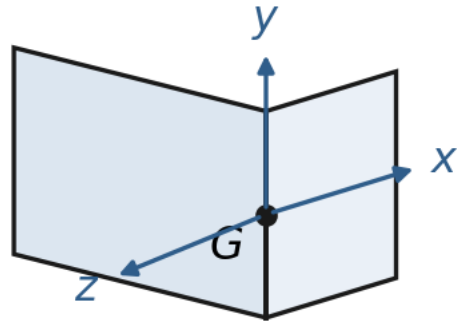
The one real trap: W is the section modulus in English and the first moment of area in French. Never copy a formula across without checking that line.

1 Internal forces: the cohesion torsor

Session 1

Cut the beam at an abscissa x and keep the left-hand part. Equilibrium of that part requires, on the cut face, a system of internal forces — the cohesion torsor — statically equivalent to a resultant and a moment at the centroid G of the section.

cut section



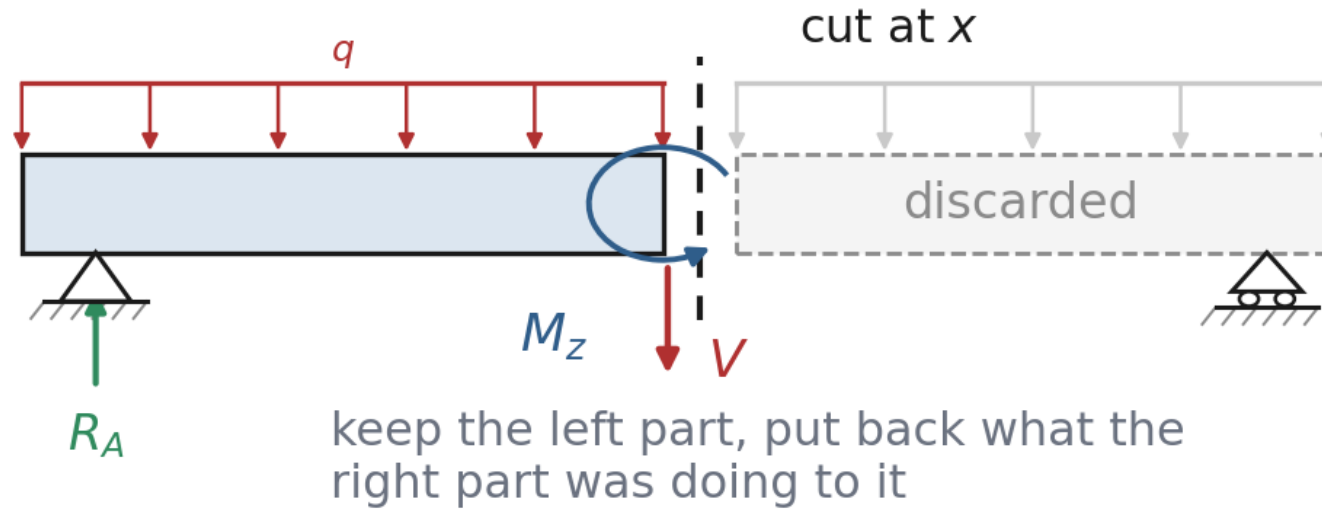
$$\{\mathcal{T}_{coh}\}_G = \begin{bmatrix} N & M_t \\ T_y & M_{fy} \\ T_z & M_{fz} \end{bmatrix}$$

N	normal force · effort normal
T_y, T_z	shear forces · efforts tranchants
M_t	torque · moment de torsion
M_{fy}, M_{fz}	bending moments · moments fléchissants

Six components in space. In plane bending only two survive: V and M_z .

Setting components to zero one at a time generates the four simple loadings — the next slide.

1 How a cut is actually taken



Three habits

- One cut per segment — a new segment starts at every load, support or discontinuity.
- Always draw the free body. Never write V and M from memory.
- Check at the far end: M_z must vanish at a free end and at a simple end support.

The recipe: (1) find the reactions (2) cut just after the point of interest (3) keep one side — usually the one with fewer forces (4) write $\sum F_y = 0$ for V and $\sum M = 0$ about the cut for M_z .

1 The four simple loadings, read off the torsor

Session 1

Tension / compression

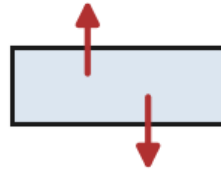


$$\begin{bmatrix} N & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\sigma = \frac{N}{A}$$

$$\sigma \leq [\sigma]$$

Shear



$$\begin{bmatrix} 0 & 0 \\ T_y & 0 \\ T_z & 0 \end{bmatrix}$$

$$\tau = \frac{T}{A}$$

$$\tau \leq [\tau]$$

Torsion

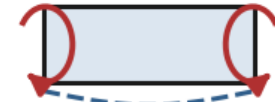


$$\begin{bmatrix} 0 & M_t \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tau = \frac{M_t \rho}{I_p}$$

$$\tau \leq [\tau]$$

Pure bending



$$\begin{bmatrix} 0 & 0 \\ 0 & M_{fy} \\ 0 & M_{fz} \end{bmatrix}$$

$$\sigma = -\frac{M_{fz}}{I_z} y$$

$$\sigma_{max} \leq [\sigma]$$

One pattern: an internal force, divided by a geometric property of the section, compared with an allowable stress.

Chapter 4 treats the combinations, where several act at once.

1 The allowable stress and the safety factor

Session 1

$$[\sigma] = \frac{\sigma_e}{s} \equiv R_{pe} = \frac{R_e}{s} \qquad [\tau] = \frac{\tau_e}{s} \equiv R_{pg} = \frac{R_{eg}}{s}$$

σ_e (or R_e) is the elastic limit of the material — a property measured in a tensile test. $s > 1$ is the safety factor — a decision, taken by the code or by the engineer, that covers the scatter of the material, the uncertainty in the loads and the imperfections of execution.

$$\text{steel S235 : } R_e = 235 \text{ MPa, } s = 1.5 \Rightarrow [\sigma] = 157 \text{ MPa} \qquad [\tau] \approx 0.6[\sigma]$$

Steel

$s \approx 1.5$ in elastic design. S235: $[\sigma] \approx 157$ MPa. Well-known material, tight tolerances, so a small factor is enough.

Concrete

$s \approx 1.5$ on the concrete and 1.15 on the steel, applied in the modern codes to the loads as well — partial factors rather than one global s .

Timber

$s \approx 2.5$ to 3, and the value depends on the duration of the load and on the moisture content. Natural material, large scatter.

1 The three kinds of bending

Session 1

Pure bending

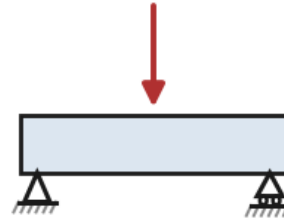


flexion pure

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & M_{fz} \end{bmatrix}$$

$V = 0, M_z = \text{const}$

Simple bending

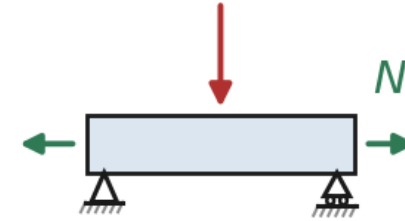


flexion simple

$$\begin{bmatrix} 0 & 0 \\ T_y & 0 \\ 0 & M_{fz} \end{bmatrix}$$

$V \neq 0, M_z \neq 0$

Plane bending



flexion plane

$$\begin{bmatrix} N & 0 \\ T_y & 0 \\ 0 & M_{fz} \end{bmatrix}$$

with an axial force N

Plane bending (flexion plane) is the geometric condition: the section has an axis of symmetry G_y , the loads act in the plane of symmetry xy , and the deflected axis stays a plane curve in that plane. This whole chapter lives inside that condition.

2

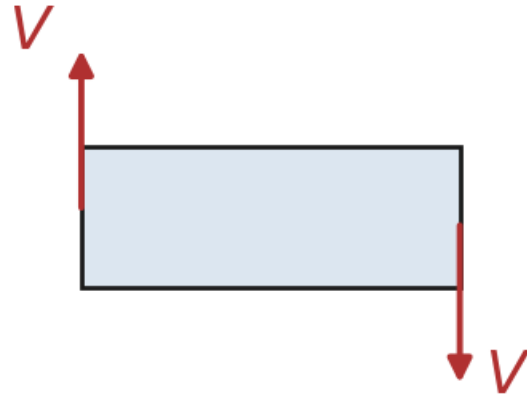
Shear force and bending moment

Sign conventions · differential relations · V and Mz diagrams

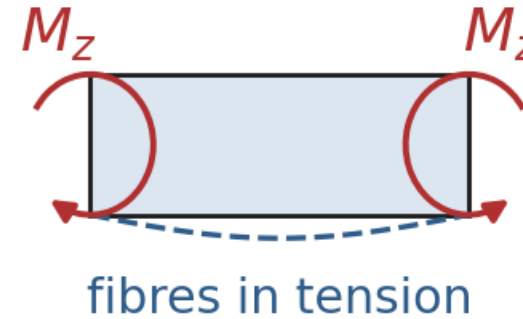
2 Sign conventions

Session 1

Positive shear force V



Positive bending moment M_z



Shear force V

Positive when it tends to rotate the isolated element clockwise.

Working from the left: V = sum of all upward external forces to the left of the cut.

Bending moment M_z

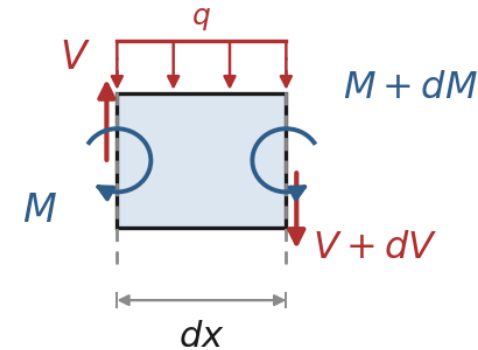
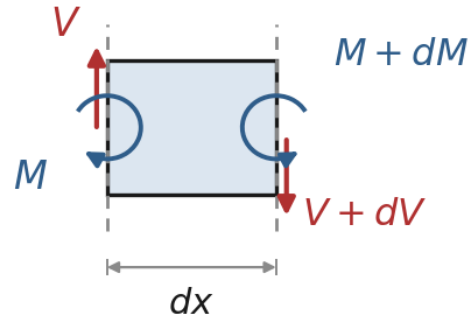
Positive when it produces sagging — tension in the lower fibres.

Working from the left: M_z = sum of the moments about the cut of the forces to the left, clockwise positive.

Isolate an element of length dx and write its equilibrium. Nothing else is needed — no material property, no hypothesis about the section.

$$\frac{dM_z}{dx} = V$$

$$\frac{dV}{dx} = -q$$



$$\sum F_y = 0: \quad V - (V + dV) - q dx = 0 \implies \frac{dV}{dx} = -q$$

$$\sum M = 0: \quad (M + dM) - M - V dx - q dx \frac{dx}{2} = 0 \implies \frac{dM_z}{dx} = V$$

The term in dx^2 is second order and disappears in the limit. Everything that follows in this section is a consequence of these two lines.

2 The three relations, and the area method

Session 1

Collected in one place, with q counted positive downwards. Differentiating the second relation once more gives the third:

$$\frac{dV}{dx} = -q(x) \quad \frac{dM_z}{dx} = V(x) \quad \frac{d^2 M_z}{dx^2} = -q(x)$$

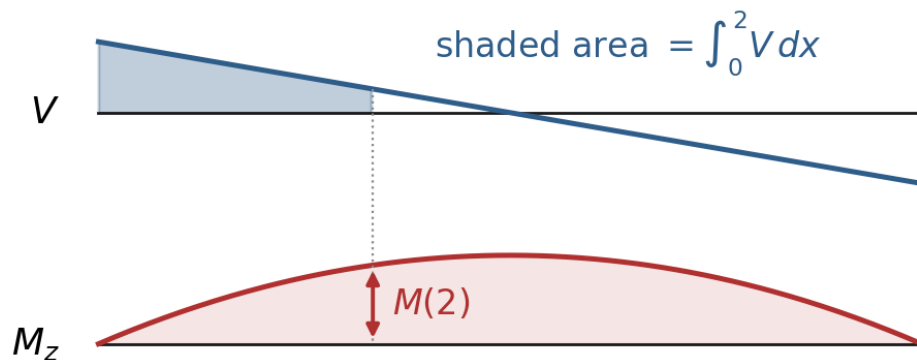
Integrating between two sections gives the area method, the fastest way to draw the diagrams by hand:

$$V_2 - V_1 = -\int_{x_1}^{x_2} q \, dx \quad M_2 - M_1 = \int_{x_1}^{x_2} V \, dx$$

The change in V over a segment = minus the area of the load diagram. **The change in M_z over a segment = the area of the shear diagram.**

2 The area method, in one picture

Session 1



the change in M_z between two sections = the area of the V diagram between them

How to use it

Start at the left end where you know V and M_z . Move right, adding the area under the curve above. You never need the equation of the segment — only its area.

Useful areas

rectangle: $b \times h$

triangle: $b \times h / 2$

parabola (2nd deg.): $2 b h / 3$

and its complement: $b h / 3$

Two consequences to keep: M_z is stationary where $V = 0$ — this is how you locate $M_{\max}$ without differentiating. And where V changes sign, M_z has a peak.

2 What the relations tell you before any number

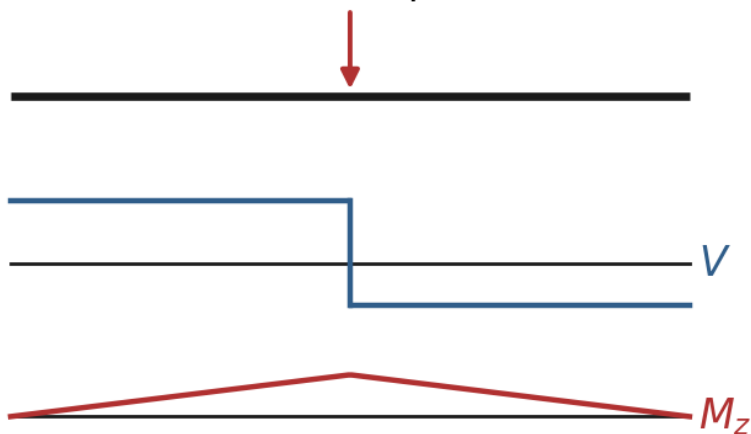
Session 1

On a segment where	$V(x)$ is	$M_z(x)$ is
$q = 0$	constant	linear
$q = \text{constant}$	linear	parabolic (2nd degree)
q linear	parabolic	cubic

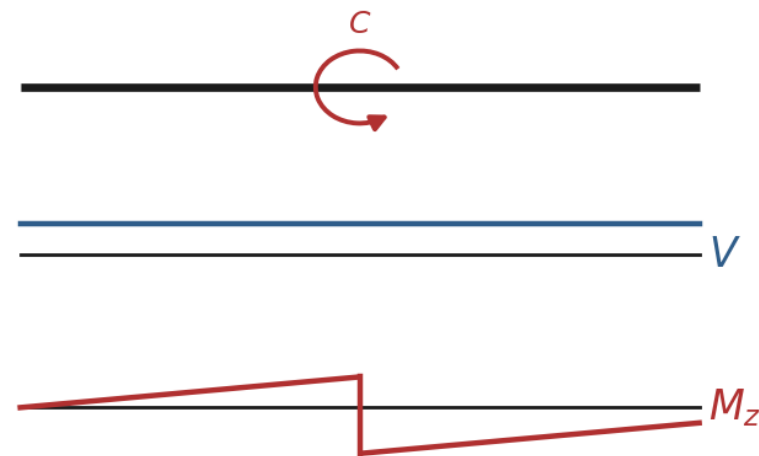
Three rules used constantly

- M_z is stationary where $V = 0$ — this locates $M_{\max}$.
- A concentrated force $\rightarrow$ step in V , kink in M_z .
- A concentrated couple $\rightarrow$ step in M_z , nothing in V .

Concentrated force: step in V , kink in M_z



Concentrated couple: step in M_z , none in V



2 A method in five steps

Session 1

1

Reactions

Free-body diagram of the whole beam. ΣM about one support, then ΣF_y . Check with ΣM about the other support.

2

Segments

Cut the beam at every load, support, and start/end of a distributed load. One segment between consecutive cuts.

3

V and Mz

In each segment, cut and write $\Sigma F_y = 0$ and $\Sigma M = 0$. Or use the area method from the left end.

4

Characteristic values

Evaluate at the ends of every segment, and where $V = 0$. Nothing else is needed.

5

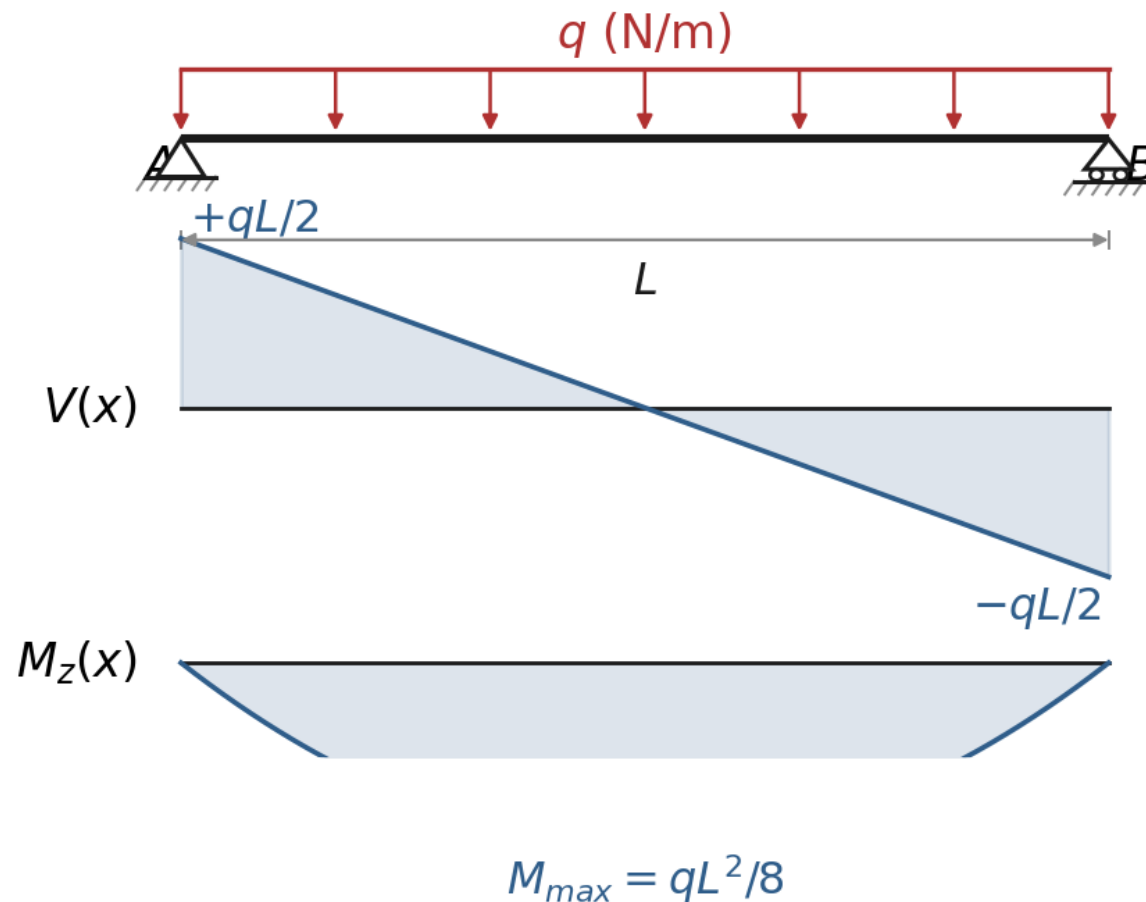
Draw and check

Shapes follow the table. $M_z = 0$ at a free end and at a simple end support. V jumps by the applied force; M_z jumps by the applied couple.

We now apply exactly this method to two reference cases and to Example A — and you will use it unchanged for the rest of the year.

2 Reference case: simple support + uniform load

Session 1



Results to know by heart

$$V(x) = q\left(\frac{L}{2} - x\right) \quad M_z(x) = \frac{qx}{2}(L - x) \quad M_{max} = \frac{qL^2}{8}$$

and, for the two other reference cases:

- central point load: $M_{max} = P L / 4$
- cantilever, uniform load: $M_{max} = q L^2 / 2$

(hogging)

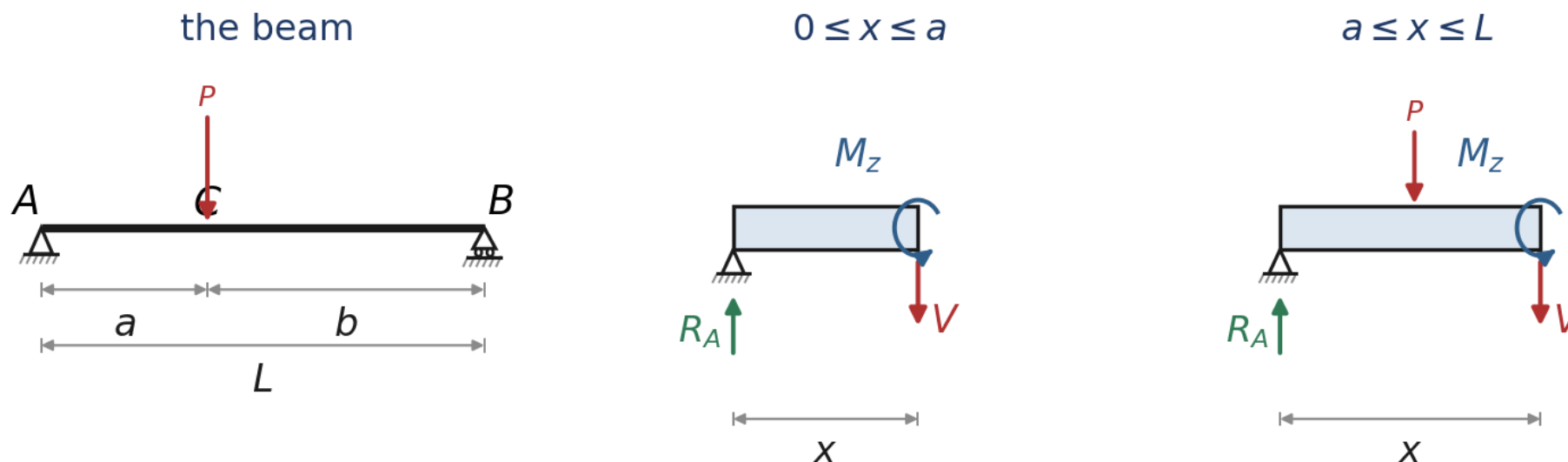
Reading the diagrams

- V is linear because q is constant.
- V changes sign at mid-span — so that is where M_z is maximum.
- M_z is parabolic and vanishes at both simple supports.
- $M_{max} = qL^2/8$: the 8 comes from the area of a triangle, $\frac{1}{2} \times L/2 \times qL/2$.

2 Second reference case: a point load anywhere on the span

Session 1

A simply supported beam AB of span L carries a single concentrated load P at C , at a distance a from A and b from B . Find V and M_z on each side of C — in symbols, once and for all.



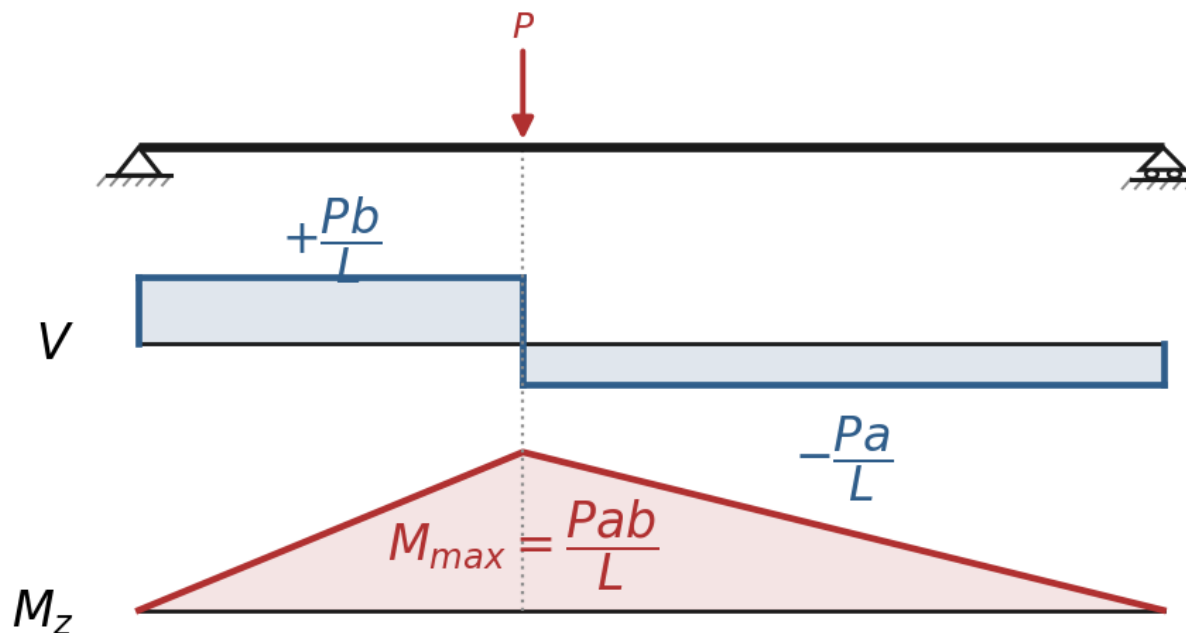
$$\sum M_A = 0: \quad R_B L = Pa \Rightarrow R_B = \frac{Pa}{L} \quad R_A = \frac{Pb}{L}$$

$$0 \leq x \leq a: \quad V = +\frac{Pb}{L} \quad M_z = \frac{Pb}{L} x$$

$$a \leq x \leq L: \quad V = -\frac{Pa}{L} \quad M_z = \frac{Pa}{L} (L - x)$$

2 Second reference case — the diagrams and Pab/L

Session 1



Read the shapes

$q = 0$ on each segment, so V is constant and M_z is linear. V steps by P at C , and M_z has a kink there — never a step, because no couple is applied.

Three cases to know by heart

central point load: $PL/4$

load anywhere: Pab/L

uniform load: $qL^2/8$

All three are simply the area under the shear diagram.

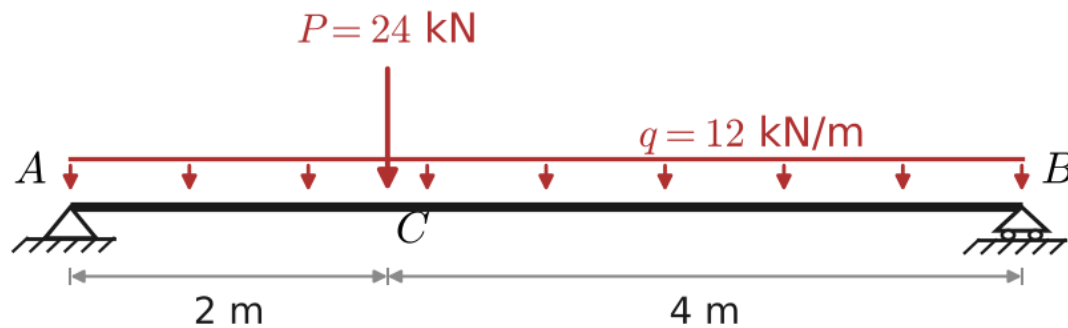
V steps by P at C ; M_z has a kink there and its maximum

$$M_{max} = M_z(a) = \frac{Pab}{L} \quad a = b = \frac{L}{2} \Rightarrow M_{max} = \frac{PL}{4}$$

The worst position for a given load is mid-span: ab is largest when $a = b$. A heavy machine belongs near a column, not in the middle of a bay.

Example A — shear force and bending moment diagrams

A simply supported beam AB of span $L = 6\text{ m}$ carries a uniform load $q = 12\text{ kN/m}$ over its whole length and a point load $P = 24\text{ kN}$ at C, 2 m from A. Draw the V and Mz diagrams and locate the maximum moment.



Two segments, because the loading changes at C — not because the beam is in two pieces.

Step 1 — the reactions, from statics alone

$$\sum M_{/B} = 0 \implies R_A = \frac{312}{6} = 52\text{ kN} \uparrow$$

$$\sum F_y = 0 \implies R_B = 96 - 52 = 44\text{ kN} \uparrow$$

Moments about B: $6 R_A = 12 \times 6 \times 3 + 24 \times 4 = 312$. Always check the sum: $52 + 44 = 96\text{ kN}$, and the total load is $12 \times 6 + 24 = 96\text{ kN}$. A reaction error poisons every diagram that follows, so spend ten seconds here.

Step 2 — decide where to cut

The expression of the loading changes at C, so the beam has two segments: $0 \leq x \leq 2$ and $2 \leq x \leq 6$. One equation per segment — never one equation across a point load.

2 Example A — the two segments, written out

Session 1

Step 3 — cut in the first segment, keep the left-hand part

$$0 \leq x \leq 2 : \quad V(x) = 52 - 12x \quad M_z(x) = 52x - 6x^2$$

Step 4 — cut in the second segment — the point load now appears

$$2 \leq x \leq 6 : \quad V(x) = 28 - 12x \quad M_z(x) = 48 + 28x - 6x^2$$

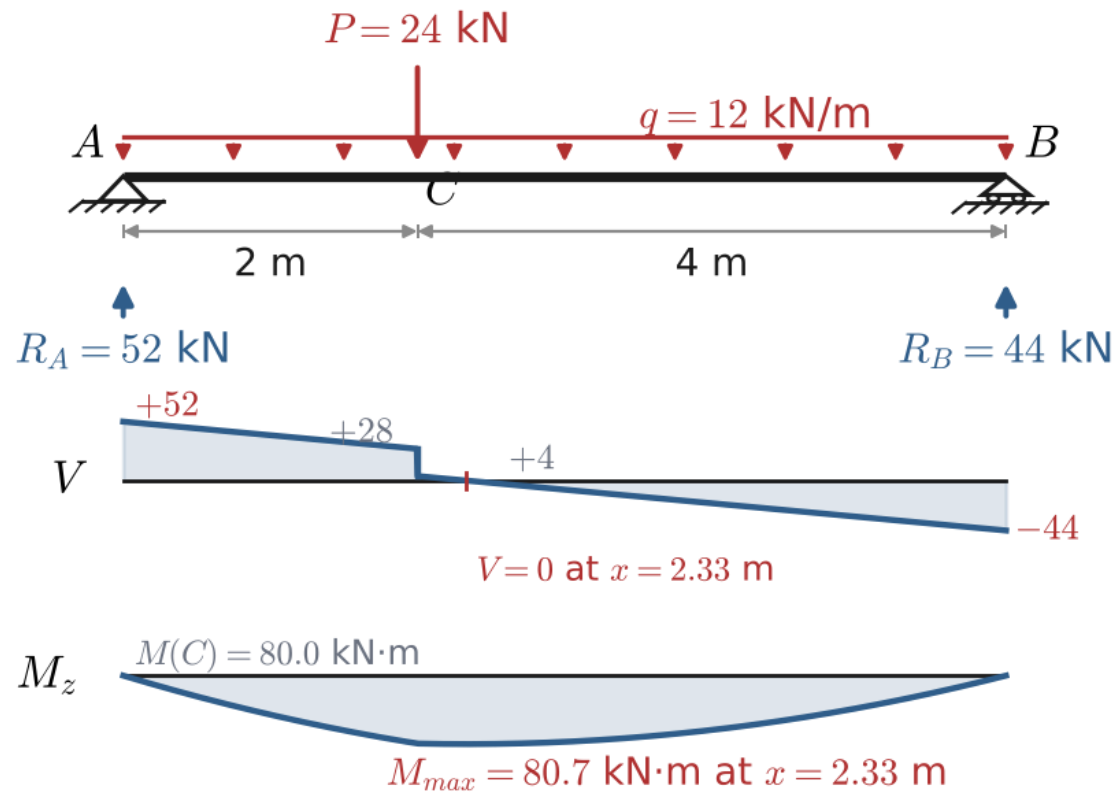
In the second segment the cut sees R_A , the uniform load over the whole length x , and the point load P . Substituting $x = 2$ in both expressions gives the same moment, 80 kN·m — the bending moment is continuous even where the shear force jumps.

Step 5 — locate the maximum moment

$$V = 0 : \quad 28 - 12x = 0 \implies x_0 = 2.33 \text{ m} \quad M_{max} = 80.7 \text{ kN} \cdot \text{m}$$

2 Example A — the diagrams, and three free checks

Session 1



$$\frac{dV}{dx} = -q \quad \frac{dM_z}{dx} = V \quad \frac{d^2M_z}{dx^2} = -q$$

Check 1 — the jump at C

V falls by exactly 24 kN at C, the value of the point load. M_z does not jump at all; only its slope changes, because $dM_z/dx = V$.

Check 2 — the area under V

Between A and C the area under V is $(52 + 28)/2 \times 2 = 80 \text{ kN}\cdot\text{m}$, and that is exactly $M(C)$. The moment diagram is the running total of the shear diagram.

Check 3 — the two ends

$M_z = 0$ at both simple supports, and M_z is maximum exactly where V crosses zero. If your diagram breaks either rule, it is wrong.

M_z is maximum where $V = 0$ — always, because $dM_z/dx = V$. Find the zero of V first, then evaluate M there.

End of Session 1 — what you must now be able to do

Session 1

$$\frac{dV}{dx} = -q \quad \frac{dM_z}{dx} = V \quad M_2 - M_1 = \int_{x_1}^{x_2} V dx$$

- ✓ **Compute reactions** — and check them with a second moment equation.
- ✓ **Identify the segments** — one per load, support or discontinuity.
- ✓ **Write V and Mz** — or accumulate areas from the left end.
- ✓ **Locate Mmax** — where $V = 0$ — or at a step in Mz.
- ✓ **Verify the diagram** — shapes, jumps and zero moment at free/simple ends.

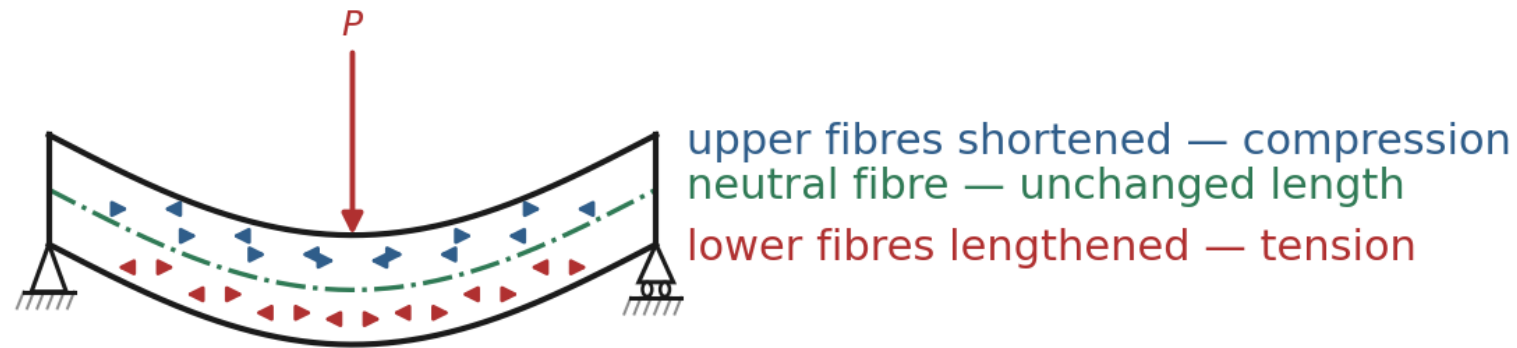
Homework before Session 2

- Redo Example A with the book closed — reactions, two segments, both diagrams.
- Exercises 1 and 2 of TD 1.
- Revise second moments of area — I and the parallel-axis theorem. We use them from the first minute of Session 2.
- Bring the section tables (IPE, HEA).

3

Normal stresses in plane bending

Session 2 · Bernoulli's hypothesis · the flexure formula · design



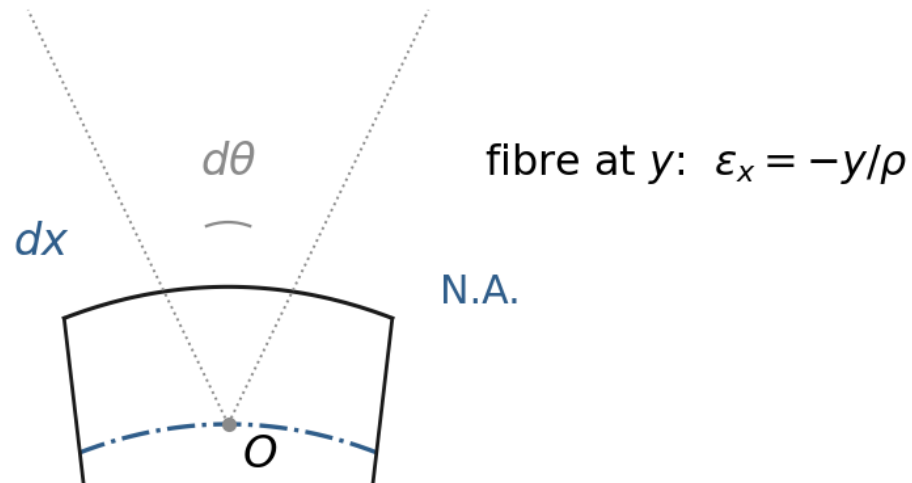
Draw two parallel lines across the side of a rubber beam and bend it. The upper fibres shorten, the lower fibres lengthen, and somewhere between them one fibre keeps its length — the neutral fibre. The shortening and the lengthening are proportional to the distance y from it.

Everything today follows from that one observation. The next three slides only turn it into arithmetic.

3 The kinematic hypothesis

Session 2

Bernoulli–Navier hypothesis: plane sections normal to the axis before deformation remain plane and normal to the deformed axis.



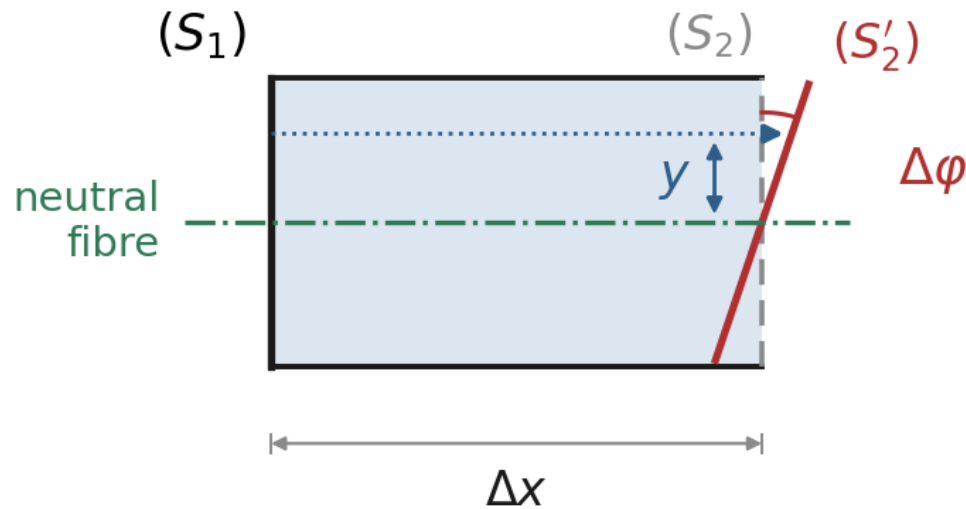
Let ρ be the radius of curvature of the neutral surface — the surface whose fibres neither lengthen nor shorten. The neutral fibre has length $dx = \rho d\theta$, and a fibre at a distance y above it has length $(\rho - y) d\theta$:

$$\epsilon_x = \frac{(\rho - y) d\theta - \rho d\theta}{\rho d\theta} = -\frac{y}{\rho}$$

The strain varies linearly with y — whatever the material. This is pure geometry, and it is why bending theory is so robust: it holds for steel, timber, concrete and plastic alike, as long as the sections stay plane.

3 The rotation of the section — $\Delta\varphi$ and the unit bending angle

Session 2



The section stays plane; it simply rotates by $\Delta\varphi$. A fibre at a height y above the neutral fibre therefore shortens by $y \Delta\varphi$:

$$\varepsilon_x = -\frac{y \Delta\varphi}{\Delta x} = -\theta y \quad \theta = \frac{\Delta\varphi}{\Delta x} = \frac{1}{\rho}$$

θ is the unit bending angle — angle unitaire de flexion — in rad/mm. It is the rotation per unit length, and therefore exactly the curvature $1/\rho$.

$$\sigma_M = E \varepsilon_x = -E \theta y = -\frac{E}{\rho} y \quad \text{and} \quad \frac{1}{\rho} = \frac{M_z}{EI_z} \implies \sigma_x = -\frac{M_z y}{I_z}$$

3 From strain to stress: the neutral axis

For a linear elastic material in uniaxial stress:

$$\sigma_x = E \varepsilon_x = -\frac{E}{\rho} y$$

In pure bending there is no resultant axial force on the section:

$$N = \int_A \sigma_x dA = -\frac{E}{\rho} \int_A y dA = 0 \implies \int_A y dA = 0$$

The first moment of area about the neutral axis vanishes: the neutral axis passes through the centroid G of the section.

The moment of the stresses about Gz must equal Mz:

$$M_z = - \int_A \sigma_x y dA = \frac{E}{\rho} \int_A y^2 dA = \frac{EI_z}{\rho}$$

which gives the curvature–moment relation:

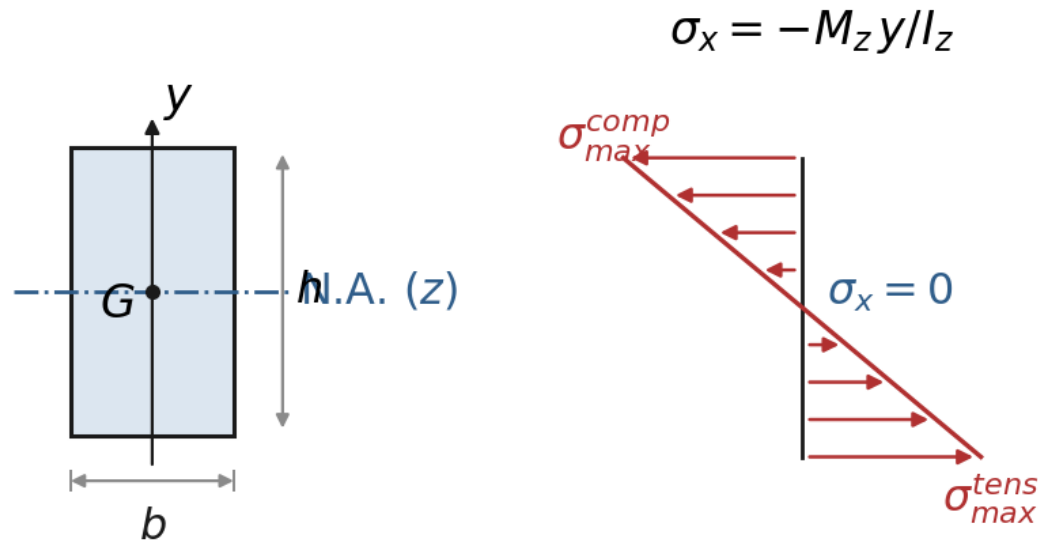
$$\frac{1}{\rho} = \frac{M_z}{EI_z}$$

EI_z is the flexural rigidity — the beam's resistance to being bent. It is the product of a material and a shape.

3 The flexure formula (Navier)

Session 2

$$\sigma_x = -\frac{M_z y}{I_z}$$



Reading the formula

- σ_x varies linearly over the depth.
- It is zero on the neutral axis and extreme at the outer fibres.
- For a sagging moment the upper fibres are compressed and the lower ones stretched — hence the minus sign, with y measured upwards.
- Many French texts write $\sigma = M y / I$ with y measured downwards: the physics is identical, only the bookkeeping differs.
- In an exam, state your convention once and keep it.

3 Section modulus and the strength condition

Session 2

$$W_z = \frac{I_z}{y_{max}} \qquad \sigma_{max} = \frac{|M_z|_{max}}{W_z} \leq [\sigma] = \frac{\sigma_e}{s}$$

Usual section moduli

$$W_{rect} = \frac{bh^2}{6} \qquad W_{circ} = \frac{\pi d^3}{32} \qquad W_{tube} = \frac{\pi(D^4 - d^4)}{32D}$$

A fuller table is given in Appendix A of the lecture notes.

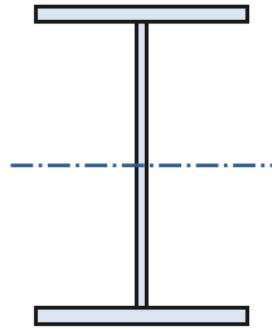
Unsymmetric sections

A T-section has two different moduli, $W_{sup} = I_z / y_{sup}$ and $W_{inf} = I_z / y_{inf}$. Both extreme fibres must be checked — essential for cast iron and concrete, which are much weaker in tension than in compression.

How a section is designed — four steps

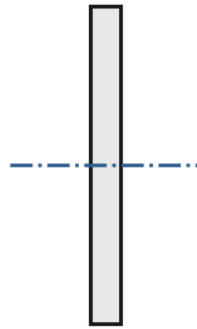
1. Compute the internal forces and find $|M_z|_{max}$.
2. Required modulus: $W_{req} = |M_z|_{max} / [\sigma]$.
3. Choose the first section in the tables with $W_z \geq W_{req}$.
4. Verify shear, then verify the deflection (Chapter 2) — which for spans over about 6 m is usually what really decides.

same weight of steel — 2.1 times the strength



welded I

$$I_z = 1.38 \times 10^8 \text{ mm}^4$$



same area, 29 × 300

$$I_z = 0.65 \times 10^8 \text{ mm}^4$$

$$I_z = \frac{BH^3 - (B - t_w)h_w^3}{12} = 1.384 \times 10^8 \text{ mm}^4 \quad W_z = 922 \text{ cm}^3$$

$$\text{same area } 29 \times 300: \quad I_z = 0.653 \times 10^8 \text{ mm}^4 \quad W_z = 435 \text{ cm}^3$$

Same steel, same weight — 2.1 times the bending strength.

Why

$I_z = \int y^2 dA$. Each element of area contributes in proportion to the square of its distance from the neutral axis. Material near the axis contributes almost nothing.

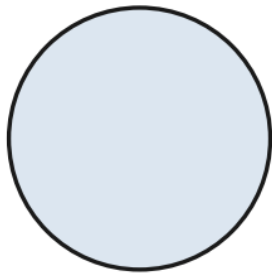
The price of the shape

Push the flanges further apart and the web becomes thin and slender: it can buckle locally. This is what limits h/t_w in the codes, and why rolled sections stop at certain proportions.

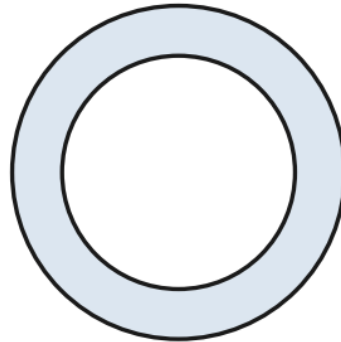
3 The same idea for a shaft — and the depth rule

Session 2

same area of metal — 2.1 times the section modulus



solid, $d = 100 \text{ mm}$
 $W = 9.8 \times 10^4 \text{ mm}^3$



tube, 140/98 mm
 $W = 2.05 \times 10^5 \text{ mm}^3$

$$W_{solid} = \frac{\pi d^3}{32} = 9.82 \times 10^4 \text{ mm}^3 \quad W_{tube} = \frac{\pi(D^4 - d^4)}{32D} = 2.05 \times 10^5 \text{ mm}^3$$

Read it as a rule: for the same amount of metal, a tube of larger diameter is always stronger in bending than a solid bar. Bicycle frames, scaffolding, masts — all tubes.

The depth rule

$$W_{rect} = \frac{bh^2}{6} \propto h^2 \quad I_{rect} = \frac{bh^3}{12} \propto h^3$$

Doubling the depth of a rectangular beam multiplies its strength by 4 and its stiffness by 8. Doubling the width multiplies both by 2 only.

Which is why...

a floor joist is laid on edge, never flat; and why the first question in any beam design is "how deep may I go?" — the answer usually comes from the architect, not from mechanics.

3 Example B — normal stresses, by Navier

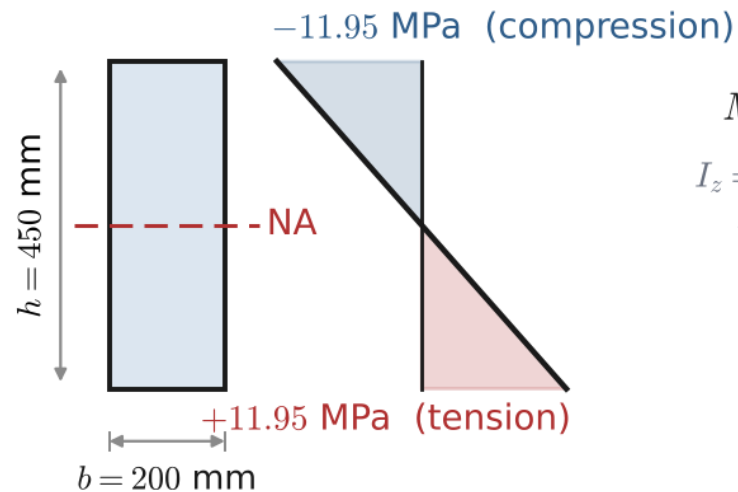
Session 2

The beam of Example A is a solid timber section, 200 mm wide and 450 mm deep, with an allowable stress $[\sigma] = 12 \text{ MPa}$. Find the largest normal stress and carry out the strength check.

Step 1 — the flexure formula, and what each symbol is

$$\sigma_x(y) = \frac{M_z}{I_z} y$$

$$\sigma_{max} = \frac{M_z}{I_z} v = \frac{M_z}{W_z}$$



$$\begin{aligned} M &= 80.7 \text{ kN}\cdot\text{m} \\ I_z &= 1.519 \times 10^{-3} \text{ m}^4 \\ W_z &= 6750 \text{ cm}^3 \end{aligned}$$

σ varies linearly with the distance y measured from the neutral axis, which passes through the centroid of the section. The two extreme fibres carry the largest stress — one in tension, one in compression — and everything between them carries less.

$W_z = I_z / v$ collapses the whole geometry of the section into a single number. It is the only section property a strength check needs.

Example B — the section properties, then the stress

Step 2 — the second moment of area

$$I_z = \frac{bh^3}{12} = \frac{0.200 \times 0.450^3}{12} = 1.519 \times 10^{-3} \text{ m}^4 = 151\,875 \text{ cm}^4$$

Step 3 — the section modulus — the single number a designer needs

$$W_z = \frac{I_z}{v} = \frac{bh^2}{6} = \frac{0.200 \times 0.450^2}{6} = 6.75 \times 10^{-3} \text{ m}^3 = 6750 \text{ cm}^3$$

Step 4 — the largest normal stress

$$\sigma_{max} = \frac{M_{max}}{W_z} = \frac{80.67}{6.75 \times 10^{-3}} = 11\,951 \text{ kPa} = 11.95 \text{ MPa}$$

Step 5 — the strength check: $\sigma_{max} = 11.95 \text{ MPa} \leq [\sigma] = 12 \text{ MPa}$. The section is adequate, with 99.6 % of the allowable stress used.

3 Example B — reading the result

Session 2

Fibre, distance y from the neutral axis	σ (MPa)	share of $\sigma_{\max}$	what it means
$y = 225$ mm — the extreme fibre	11.95	100 %	sizes the beam
$y = 150$ mm	7.97	67 %	two thirds of the depth
$y = 100$ mm	5.31	44 %	lightly stressed
$y = 50$ mm	2.66	22 %	almost idle
$y = 0$ — the neutral axis	0	0 %	carries no bending stress

Where the material works

Half the depth of this beam carries less than a quarter of the maximum stress. The material near the neutral axis is there to carry shear and to hold the flanges apart, not to resist bending.

Why depth counts twice

$W_z = bh^2/6$: doubling the width doubles the strength, doubling the depth multiplies it by four. Deep and narrow always beats wide and shallow — until the beam buckles sideways.

If the check had failed

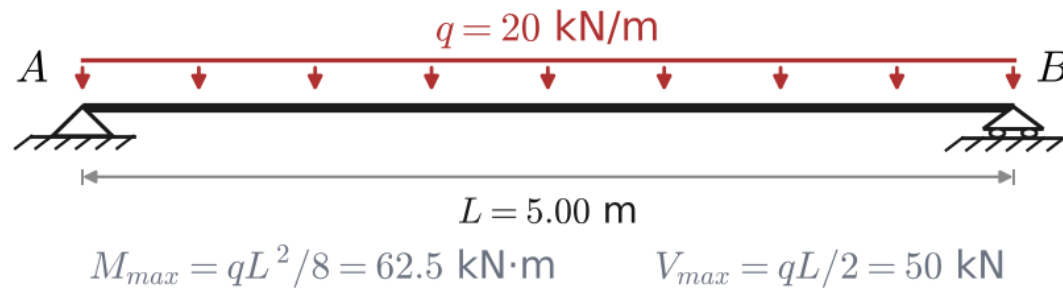
Three honest remedies: a deeper section, a better grade of timber, or a shorter span. A wider section is the expensive one, because width only buys strength in proportion.

An I-section is this table turned into a shape: metal taken away from where the stress is low and put where the stress is high.

3 Application — choosing a steel section

Session 2

A steel floor beam spans $L = 5\text{ m}$ and carries a uniform load $q = 20\text{ kN/m}$. Take $[\sigma] = 160\text{ MPa}$. Choose an IPE section, then check that the web is not overloaded in shear.



Design is the formula read backwards: the strength condition is solved for the section property, then a catalogue supplies the section.

Step 1 — the internal forces that matter

$$M_{max} = \frac{qL^2}{8} = \frac{20 \times 25}{8} = 62.5\text{ kN}\cdot\text{m} \quad V_{max} = \frac{qL}{2} = 50\text{ kN}$$

A uniform load on a simply supported beam puts the largest moment at mid-span and the largest shear at the supports — the two never occur at the same section, which is why they are checked separately.

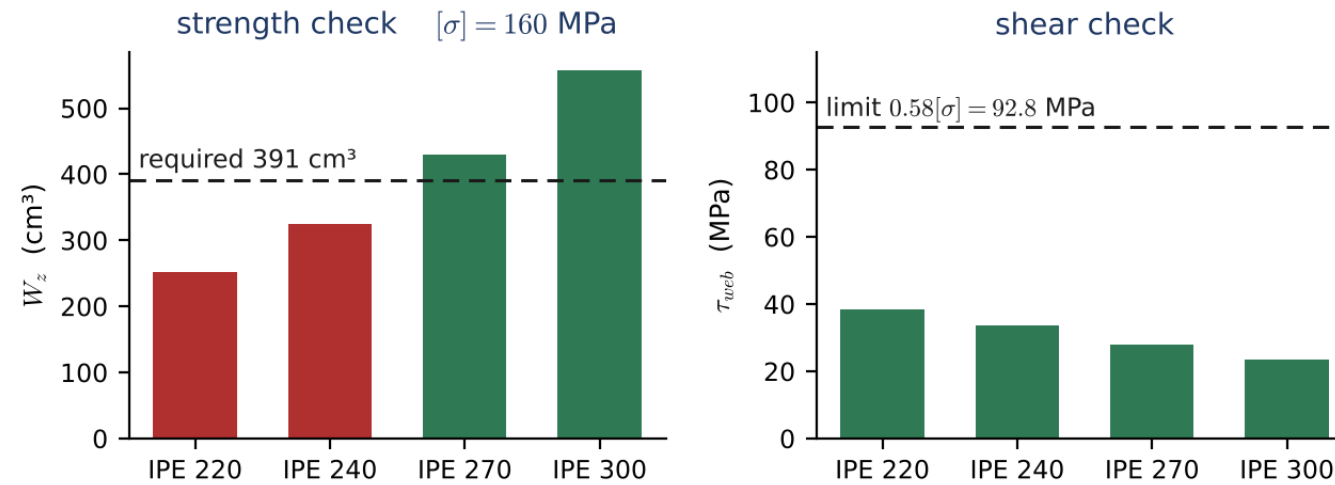
Step 2 — the section modulus required

$$W_{req} = \frac{M_{max}}{[\sigma]} = \frac{62.5 \times 10^3}{160 \times 10^6} = 3.91 \times 10^{-4}\text{ m}^3 = 391\text{ cm}^3$$

Running down the table: IPE 220 offers 252 cm^3 and IPE 240 offers 324 cm^3 — both fail. IPE 270, with 429 cm^3 , is the first adequate section.

3 Application — the two checks side by side

Session 2



Step 3 — the bending stress in the chosen section

$$\text{IPE 270 : } W_z = 429 \text{ cm}^3 \Rightarrow \sigma = \frac{62.5}{429 \times 10^{-6}} = 145.7 \text{ MPa} \leq 160$$

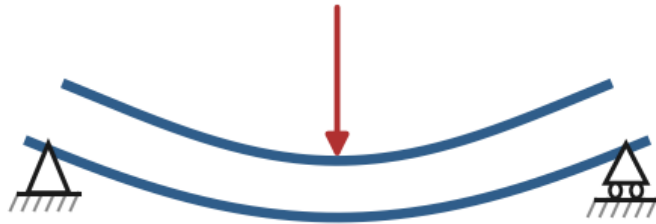
Step 4 — the shear stress in the web

$$\tau_{web} \simeq \frac{V}{h t_w} = \frac{50}{0.270 \times 0.0066} = 28.1 \text{ MPa} \leq 0.58 [\sigma] = 92.8 \text{ MPa}$$

4

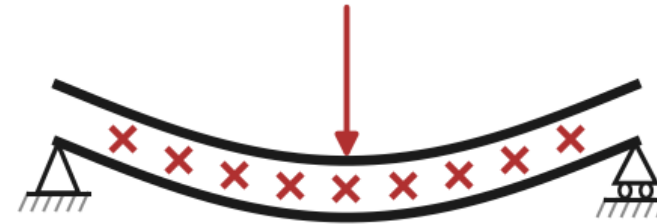
Shear stresses in plane bending

Jouravski's formula · distribution in usual sections · shear flow



not connected: the planks slide

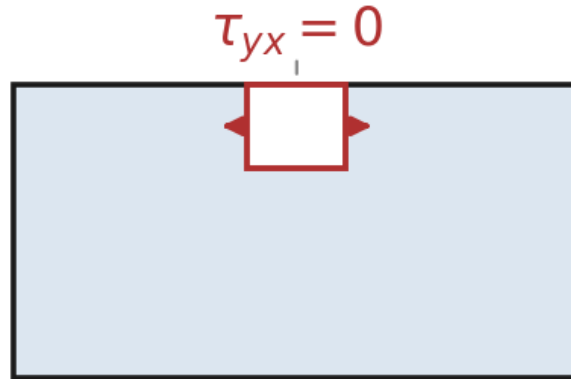
Stack two identical planks and bend them: they slide over one another at the interface, and the ends no longer line up. Glue them and the sliding is prevented — by shear stresses transmitted through the glue.



connected: shear stresses appear

In a solid beam the same shear stresses act on every horizontal plane; and by the theorem of conjugate shear stresses, $\tau_{xy} = \tau_{yx}$, they act on the cross-section as well. Two glued planks are 4 times stiffer and 2 times stronger than two loose ones.

top face of the beam — nothing pushes sideways on it



$$\tau_{xy} = \tau_{yx} \Rightarrow \tau_{xy} = 0 \text{ there too}$$

so the parabola must start from zero at the top and bottom fibres

The argument in three steps

1. Nothing touches the top face, so no shear stress can act on it: $\tau_{yx} = 0$ there.
2. Shear stresses are conjugate: $\tau_{xy} = \tau_{yx}$.
3. Therefore τ_{xy} — the shear stress on the cross-section — is also zero at the top and bottom fibres.

Consequence: σ and τ never reach their maxima at the same point of a rectangular section. The outer fibres carry $\sigma_{\max}$ with no shear; the neutral axis carries $\tau_{\max}$ with no normal stress.

4 Jouravski's formula (1855)

Isolate the portion of the element dx lying above a horizontal plane at the level y . The normal stresses on its two end faces differ, because Mz differs; equilibrium along x then requires a shear force $\tau t dx$ on the horizontal face:

$$\tau = \frac{VS_z^*}{I_z t}$$

$$S_z^* = \int_{A^*} \eta dA = A^* \eta_G^*$$

S_z^* is the first moment, about the neutral axis, of the area lying beyond the level considered; t is the width of the section at that level. the same formula reads: $\tau = \frac{VS_z^*}{I_z t} \equiv \tau = \frac{T_y W_{Gz}}{b I_{Gz}}$

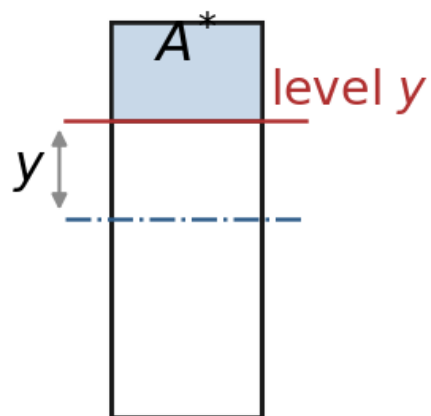
S_z^* may be computed from the area above or below the level — the two give the same magnitude.

t is the width at the level where τ is wanted, not the maximum width.

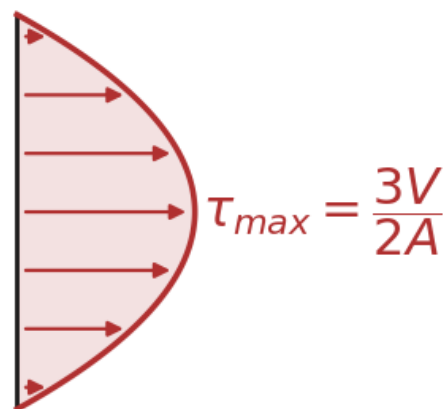
τ is largest where S_z^* / t is largest — for compact sections, at the neutral axis.

4 Distribution in a rectangular section

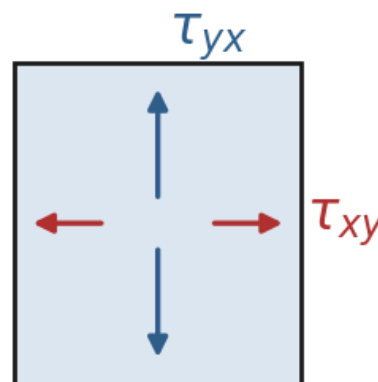
First moment S_z^*



Parabolic τ_{xy}



$\tau_{xy} = \tau_{yx}$



$$\tau(y) = \frac{6V}{bh^3} \left(\frac{h^2}{4} - y^2 \right) \quad \tau_{max} = \frac{3V}{2A} = 1.5 \tau_{mean}$$

$\tau = 0$ at the top and bottom faces, because no shear stress can act on a free surface.

Other compact sections

Solid circle: $\tau_{max} = 4V / 3A \approx 1.33 \tau_{mean}$

Thin tube: $\tau_{max} = 2V / A$

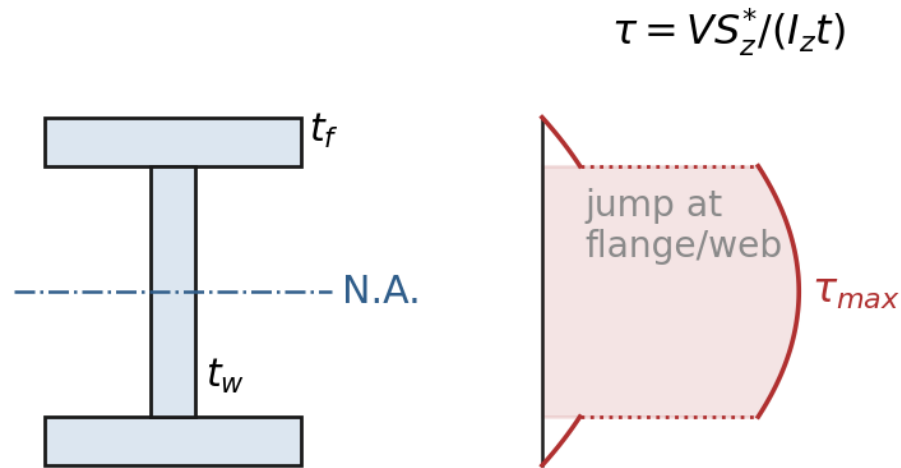
Rectangle: $\tau_{max} = 1.5 \tau_{mean}$

A parabola, not a constant

τ vanishes at the extreme fibres — where σ is largest — and is maximum on the neutral axis, where σ is zero. Using V/A instead of $1.5 V/A$ underestimates the peak by one third.

4 The I-section, and shear flow

Session 2



Almost all the shear is in the web

The jump in t at the flange/web junction produces a jump in τ in the ratio b / t_w . A rapid and safe design estimate is:

$$\tau_{max} \approx \frac{V}{h t_w}$$

Shear flow and built-up members

For planks nailed or screwed together, or plates welded to form a girder, the quantity of interest is the shear force per unit length transmitted at the junction:

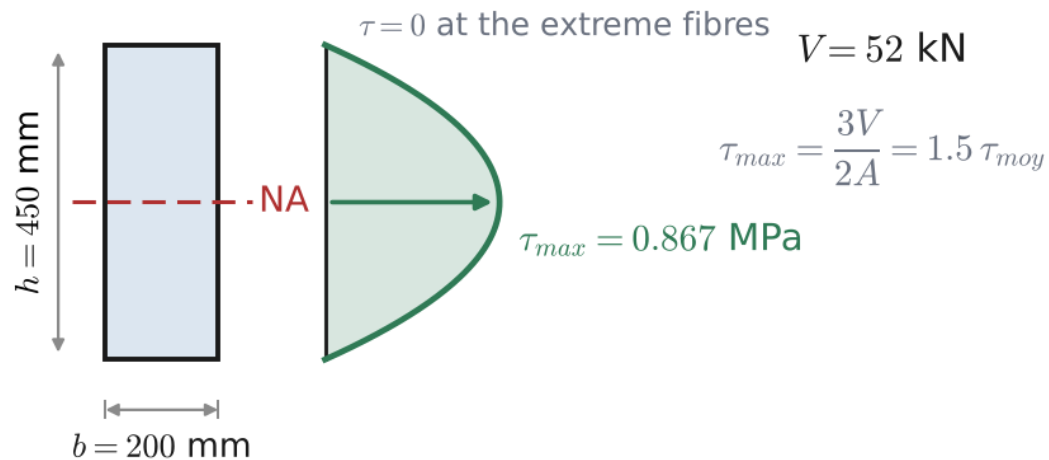
$$f = \tau t = \frac{VS_z^*}{I_z} \quad e \leq \frac{n F_{adm} I_z}{VS_z^*}$$

f follows the shear diagram, so the connectors may be spaced more closely near the supports and more widely near mid-span.

The same beam and the same 200×450 timber section. The largest shear force is $V = 52$ kN, at the support A. Find how τ is distributed over the depth and where it is largest.

Step 1 — Jouravski's formula

$$\tau(y) = \frac{VS_z^*(y)}{I_z b(y)}$$



$S^*(y)$ is the first moment, about the neutral axis, of the part of the section lying beyond the level y — the piece that would slide off if the beam were cut along that level. $b(y)$ is the width of the section at that same level. Both change as you move down the section, and that is what makes τ vary.

At the extreme fibres nothing lies beyond the level, so $S^* = 0$ and $\tau = 0$. At the neutral axis S^* is largest — the exact opposite of σ .

4 Example C — the first moment, then the maximum

Session 2

Step 2 — the first moment of the half-section, about the neutral axis

$$S_z^*(0) = b \cdot \frac{h}{2} \cdot \frac{h}{4} = \frac{bh^2}{8} = 5.063 \times 10^{-3} \text{ m}^3$$

The half-section above the neutral axis has area $b \cdot h/2$ and its own centroid sits $h/4$ above the axis. Multiply the two and you have S^* at the neutral axis — no integration needed for a rectangle.

Step 3 — the maximum shear stress

$$\tau_{max} = \frac{52 \times 5.063 \times 10^{-3}}{1.519 \times 10^{-3} \times 0.200} = 867 \text{ kPa} = 0.867 \text{ MPa}$$

Step 4 — the shape of the distribution

$$\tau(y) = \frac{3V}{2A} \left[1 - \left(\frac{2y}{h} \right)^2 \right] \quad \tau_{max} = \frac{3V}{2A} = 1.5 \tau_{moy}$$

Level in the section	$S^* (10^{-3} \text{ m}^3)$	τ (MPa)	share of $\tau_{\max}$	σ at the same level
$y = 0$ — the neutral axis	5.063	0.867	100 %	0
$y = h/4 = 112.5 \text{ mm}$	3.797	0.650	75 %	5.97 MPa
$y = h/2 = 225 \text{ mm}$ — extreme fibre	0	0	0 %	11.95 MPa

$$\frac{\tau_{\max}}{\sigma_{\max}} = \frac{0.867}{11.95} = 7.3 \%$$

Why shear is small here

Normal stress grows with the span, shear stress does not: σ goes as qL^2/W while τ goes as qL/A . Double the span and σ quadruples while τ only doubles.

When shear does matter

Short and heavily loaded beams, a large load very near a support, slender webs that can buckle, and timber or glued beams, which are weak along the grain or the glue line.

The complementary stress

τ on a vertical face is matched by an equal τ on the horizontal face. That is why a stack of unglued planks slides, and why the glue line of a built-up beam is the thing to check.

σ is largest where τ is zero, and τ is largest where σ is zero. The two checks never collide at the same fibre.

Seven mistakes that cost marks

Session 2



Forgetting a segment

A new segment starts at every load, support and end of a distributed load.



Taking $M_{\max}$ under the load

It is at the point of zero shear — or at a step in M_z .



Ignoring the sign of M_z

Hogging over a support needs reinforcement on top. The sign is the design.



Using V/A for the shear stress

The peak is $1.5 V/A$ in a rectangle, $2 V/A$ in a thin tube.



Using the wrong t in Jouravski

t is the width at the level examined, not the flange width.



Checking only one fibre of a T

$W_{\text{sup}} \neq W_{\text{inf}}$. Both must be verified.



Stopping at the strength check

The deflection check (Chapter 2) often governs spans above 6 m.

Summary of Chapter 1

$$\frac{dV}{dx} = -q \quad \frac{dM_z}{dx} = V \quad \frac{1}{\rho} = \frac{M_z}{EI_z}$$

$$\sigma_x = -\frac{M_z y}{I_z} \quad \sigma_{max} = \frac{|M_z|}{W_z} \leq [\sigma] \quad \tau = \frac{VS_z^*}{I_z t} \quad f = \frac{VS_z^*}{I_z}$$

1

The neutral axis passes through the centroid

It is the consequence of zero axial force, and the first thing to compute for any section.

2

σ is linear, τ is parabolic

Normal stresses are extreme at the outer fibres; shear stresses at the neutral axis. They never peak together.

3

Bending governs slender beams

Check shear explicitly for short, deep or thin-walled members — and for built-up sections, check the connectors.

Formulary — the standard cases

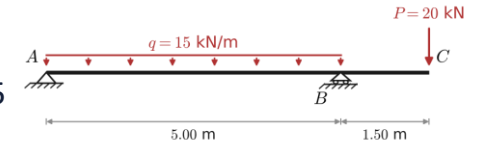
Load case	Reactions	Maximum bending moment	Max. deflection
Simply supported, uniform load q	$qL / 2$	$qL^2 / 8$ at mid-span	$5qL^4 / 384EI$
Simply supported, central load P	$P / 2$	$PL / 4$ at mid-span	$PL^3 / 48EI$
Simply supported, load P at a, b	Pb/L and Pa/L	Pab / L under the load	$\approx Pa^2b^2 / 3LEI$
Simply supported, end couple C	C / L	C at the loaded end	$CL^2 / 16EI$
Cantilever, uniform load q	qL	$-qL^2 / 2$ at the support	$qL^4 / 8EI$
Cantilever, end load P	P	$-PL$ at the support	$PL^3 / 3EI$
Propped cantilever, uniform load q	$5qL/8$ and $3qL/8$	$-qL^2 / 8$ at the fixed end	$qL^4 / 185EI$
Built-in both ends, uniform load q	$qL / 2$	$-qL^2 / 12$ ends, $+qL^2 / 24$ span	$qL^4 / 384EI$

Deflections are quoted here for completeness; they are derived in Chapter 2. A minus sign on the moment means hogging — tension on the upper fibres.

Tutorial sheet TD 1 bis — one exercise per method

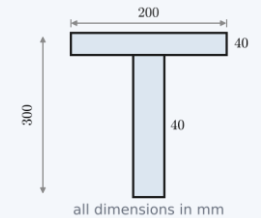
1 Shear force and bending moment diagrams

An overhanging beam ABC is supported at A and B with $AB = 5\text{ m}$, and overhangs by $BC = 1.50\text{ m}$. It carries $q = 15\text{ kN/m}$ on AB and a point load $P = 20\text{ kN}$ at the free end C. Draw the V and Mz diagrams, and give the largest moment of each sign.



2 Normal stresses — Navier

The same beam has the T-section drawn opposite: a flange $200 \times 40\text{ mm}$ on top and a web $40 \times 260\text{ mm}$ below. Locate the neutral axis, compute I_z , and find the largest tensile and compressive stresses under each of the two moments of Ex. 1.



3 Shear stresses — Jouravski

Same section, $V = 43.5\text{ kN}$. Find τ at the neutral axis and on both sides of the flange–web junction, sketch the distribution over the depth, and say why it jumps where the width changes.

*same section
as Exercise 2*

Each exercise must be solved by the method named in its title. A bilingual statement sheet and a fully detailed model solution are provided separately.

End of Chapter 1

Next: Chapter 2 — Deflection of symmetric beams in plane bending

Initial parameters · moment-area theorems · superposition · the serviceability check