



Chapter II: Laplace Transform

II.1 Introduction:

The Laplace transform is a mathematical tool that is widely used in automatic control and regulation. It is especially useful when a system is described by a differential equation. The main idea is to move from the time domain to the Laplace domain. In this new domain, differential equations become algebraic equations, which are usually easier to solve and analyze.

For a function $f(t)$, the Laplace transform is written as:

$$F(p) = L\{f(t)\}$$

And is defined by:

$$F(p) = \int_0^{+\infty} f(t)e^{-pt} dt$$

Here, p is the Laplace variable.

II.2 Properties:

II.2.1 Linearity Property:

The Laplace transform is a linear operator. This means that the transform of a sum of functions can be calculated separately.

If: $h(t) = af(t) + bg(t)$ then: $L\{h(t)\} = a L\{f(t)\} + b L\{g(t)\}$

Therefore: $L\{a f(t) + b g(t)\} = a F(p) + b G(p)$

Where a and b are constants. This property is often useful when a signal is made up of several different components.

II.2.2 Laplace Transform of a Derivative:

The derivative property is one of the most important properties in control engineering.

For the first derivative:

$$L\{f'(t)\} = pF(p) - f(0^+)$$

If the initial condition is zero, this expression becomes much simpler:

$$L\{f'(t)\} = pF(p)$$

For the second derivative:

$$L\{f''(t)\} = p^2F(p) - p f(0^+) - f'(0^+) = p^2F(p)$$

This is very useful because it allows us to transform a differential equation into an algebraic equation.

II.2.3 Inverse Laplace transform:

Sometimes we know the function in the Laplace domain and we need to find the corresponding function in the time domain.

In this case, we use the inverse Laplace transform:

$$f(t) = L^{-1}\{F(p)\}$$

In control systems, the inverse transform is often obtained using a table of common transforms or by decomposing the function into simpler terms.

II.2.4 Time translation:

A time delay can be represented easily using the Laplace transform. For a delayed function:

$$L\{f(t - a)u(t - a)\} = e^{-ap}F(p)$$

Where $u(t - a)$ is the unit step function.

This result is important when studying systems that have a time delay. A delay of a seconds in the time domain appears as the factor e^{-ap} in the Laplace domain.

II.2.5 Translation in the Laplace Variable:

Another useful property concerns multiplication by an exponential function. For example:

$$L\{e^{-at}f(t)\} = F(p + a)$$

Similarly:

$$L\{e^{at}f(t)\} = F(p - a)$$

This property allows us to obtain a new transform without calculating the integral again.

II.2.6 Change of Scale:

If the time variable is multiplied by a positive constant a , we have:

$$L\{f(at)\} = \left(\frac{1}{a}\right)F\left(\frac{p}{a}\right)$$

This property is known as the scaling property. It shows how a change in the time scale affects the corresponding Laplace transform.

II.2.7 Convolution Property:

The convolution of two functions $f(t)$ and $g(t)$ is defined by:

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau$$

One important advantage of the Laplace transform is that it changes convolution into simple multiplication:

$$L\{f(t) * g(t)\} = F(p) G(p)$$

This property is particularly useful in automatic control. For a system with an input $U(p)$ and a transfer function $G(p)$, the output can be written as:

$$Y(p) = G(p)U(p)$$

This is one of the basic relations used when studying control systems.

II.3 Common Laplace Transforms:

Several simple functions appear frequently in regulation and control. It is therefore useful to know their Laplace transforms.

✎ Unit Step: For the unit step function $u(t) = 1$, for $t \geq 0$ we have:

$$L\{u(t)\} = 1/p$$

The unit step is commonly used as an input when studying the response of a system.

✎ Exponential Function: For $f(t) = e^{-at}$ the transform is:

$$L\{e^{-at}\} = \frac{1}{(p + a)}$$

For a positive exponential:

$$L\{e^{at}\} = \frac{1}{(p - a)}$$

With the appropriate convergence condition.

✎ Complex Exponential: For $f(t) = e^{iat}$ we obtain:

$$L\{e^{iat}\} = \frac{1}{(p - ia)}$$

Where i is the imaginary unit.

✎ Cosine: For $f(t) = \cos(at)$ the Laplace transform is:

$$L\{\cos(at)\} = \frac{p}{(p^2 + a^2)}$$

✎ Sine: For $f(t) = \sin(at)$ we have:

$$L\{\sin(at)\} = \frac{a}{(p^2 + a^2)}$$

✎ Dirac Impulse: The Dirac impulse is represented by $\delta(t)$, its Laplace transform is:

$$L\{\delta(t)\} = 1$$

The impulse is important when studying the impulse response of a system.

II.4 Application to Regulation and Control:

The Laplace transform is particularly useful in Regulation and control because it helps us describe the behavior of dynamic systems in a simple way.

Consider a system with an input $u(t)$ and an output $y(t)$. After applying the Laplace transform, and assuming zero initial conditions, we can write:

$$Y(p) = G(p)U(p)$$

The transfer function is then:

$$G(p) = \frac{Y(p)}{U(p)}$$

The transfer function gives the relationship between the input and the output of the system in the Laplace domain.

For this reason, the Laplace transform is used when studying important concepts such as:

The transfer function, system response, poles and zeros, stability, transient response and steady-state response.

II.5 Exercices:

Exercise 1

Find the Laplace transform of the following functions:

1. $f(t) = 1$
2. $f(t) = t$
3. $f(t) = e^{-2t}$
4. $f(t) = \sin(3t)$
5. $f(t) = \cos(4t)$
6. $f(t) = \delta(t)$
7. $f(t) = u(t - 2)$
8. $f(t) = (t - 3)u(t - 3)$

Exercise 2

Using the properties of the Laplace transform, calculate:

1. $f(t) = 3e^{-2t} + 5 \sin(2t)$

2. $f(t) = 4\cos(3t) - 2e^{-t}$
3. $f(t) = 2t + 3e^{-4t}$

Exercise 3

Find the inverse Laplace transform of:

1. $F(p) = \frac{1}{p+2}$
2. $F(p) = \frac{3}{p^2+9}$
3. $F(p) = \frac{p}{p^2+16}$
4. $F(p) = \frac{5}{p(p+5)}$
5. $F(p) = \frac{2p+5}{(p+1)(p+3)}$

For the last two questions, use partial fraction decomposition.

Exercise 4

When $f_1(t) = t^2$ and $f_2(t) = e^{-2t}$ Find the Laplace transform of the following functions:

$$L\left\{\frac{df_1(t)}{dt}\right\} \text{ and } L\left\{\frac{df_2(t)}{dt}\right\}$$

Consider the differential equation: $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0$, with: $y(0) = 0, y'(0) = 0$

1. Apply the Laplace transform to the equation and write the resulting equation in terms of $Y(p)$.
2. If $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = u(t)$ Determine the transfer function $G(p)$.
3. Find the output for a unit-step input.
4. Determine the inverse Laplace transform.
5. Identify the poles of the system.
6. Comment on the behavior of the system.