

Chapter 2 — Set Theory

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1 Introduction

Set theory provides a fundamental language for mathematics. It allows us to describe collections of objects, study relationships between them, and construct mathematical structures.

Throughout this chapter, we use the convention

$$\mathbb{N} = \{0, 1, 2, \dots\}.$$

1.1 Sets and membership

A set is a collection of objects called its elements.

If an object x belongs to a set E , we write

$$x \in E.$$

Otherwise, we write

$$x \notin E.$$

A set may be described by listing its elements:

$$A = \{1, 2, 3, 4\},$$

or by specifying a property within a given set:

$$A = \{x \in \mathbb{N} \mid 1 \leq x \leq 4\}.$$

The order of the elements and repeated occurrences do not change a set. For example,

$$\{1, 2, 3\} = \{3, 2, 1\} = \{1, 1, 2, 3\}.$$

Remark 1.1. A set may be an element of another set. For example,

$$\{1, 2\} \in \{\{1, 2\}, \{3\}\}.$$

2 Inclusion, Equality, and the Empty Set

Definition 2.1 (Subset). Let A and B be sets. We say that A is a subset of B if every element of A belongs to B :

$$A \subseteq B \iff \forall x, \quad x \in A \Rightarrow x \in B.$$

If $A \subseteq B$ and $A \neq B$, we write

$$A \subsetneq B$$

and say that A is a proper subset of B .

Definition 2.2 (Equality). Two sets are equal if they have exactly the same elements:

$$A = B \iff \forall x, \quad x \in A \iff x \in B.$$

Proposition 2.3 (Double inclusion). For any sets A and B ,

$$A = B \iff A \subseteq B \text{ and } B \subseteq A.$$

Definition 2.4 (Empty set). The empty set, denoted by $\emptyset$ or $\varnothing$, is the set with no elements.

For every set E ,

$$\emptyset \subseteq E.$$

Proposition 2.5. *For all sets A , B , and C :*

1. $A \subseteq A$;
2. if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$;
3. if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

3 Operations on Sets

Definition 3.1 (Union). The union of A and B is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Definition 3.2 (Intersection). The intersection of A and B is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Two sets are disjoint if their intersection is empty.

Definition 3.3 (Difference). The difference of A and B is

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Definition 3.4 (Symmetric difference). The symmetric difference of A and B is

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

Equivalently,

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

Example 3.5. If

$$A = \{1, 2, 3\}, \quad B = \{3, 4, 5\},$$

then

$$A \cup B = \{1, 2, 3, 4, 5\},$$

$$A \cap B = \{3\},$$

$$A \setminus B = \{1, 2\},$$

$$B \setminus A = \{4, 5\},$$

$$A \triangle B = \{1, 2, 4, 5\}.$$

3.1 Basic identities

For arbitrary sets A , B , and C :

Identity and idempotence

$$A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset,$$

$$A \cup A = A, \quad A \cap A = A.$$

Commutativity

$$A \cup B = B \cup A, \quad A \cap B = B \cap A.$$

Associativity

$$(A \cup B) \cup C = A \cup (B \cup C),$$

$$(A \cap B) \cap C = A \cap (B \cap C).$$

Distributivity

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

3.2 Indexed unions and intersections

Let $(A_i)_{i \in I}$ be a family of sets. Its union is

$$\bigcup_{i \in I} A_i = \{x \mid \text{there exists } i \in I \text{ with } x \in A_i\}.$$

If I is nonempty, its intersection is

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for every } i \in I\}.$$

When all sets are regarded as subsets of a fixed set E , the empty intersection is conventionally taken to be E .

4 Power Sets and Complements

Definition 4.1 (Power set). The power set of E is the set of all subsets of E :

$$\mathcal{P}(E) = \{A \mid A \subseteq E\}.$$

Thus,

$$A \in \mathcal{P}(E) \iff A \subseteq E.$$

In particular,

$$\emptyset \in \mathcal{P}(E), \quad E \in \mathcal{P}(E).$$

Example 4.2. If $E = \{a, b\}$, then

$$\mathcal{P}(E) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Proposition 4.3. If $|E| = n$ is finite, then

$$|\mathcal{P}(E)| = 2^n.$$

Proof. Each element of E may either be included in a subset or excluded from it. There are two independent choices for each of the n elements, giving 2^n subsets. $\square$

Definition 4.4 (Complement). Let $A \subseteq E$. The complement of A in E is

$$A^c = E \setminus A.$$

The notation A^c assumes that the reference set E has been fixed.

The following identities hold:

$$\begin{aligned} \emptyset^c &= E, & E^c &= \emptyset, & (A^c)^c &= A, \\ A \cup A^c &= E, & A \cap A^c &= \emptyset. \end{aligned}$$

Theorem 4.5 (De Morgan's laws). For $A, B \subseteq E$,

$$\begin{aligned} (A \cup B)^c &= A^c \cap B^c, \\ (A \cap B)^c &= A^c \cup B^c. \end{aligned}$$

Proof. For every $x \in E$,

$$\begin{aligned} x \in (A \cup B)^c &\iff x \notin A \cup B \\ &\iff x \notin A \text{ and } x \notin B \\ &\iff x \in A^c \cap B^c. \end{aligned}$$

The second identity follows similarly by replacing “and” with “or”. □

5 Cartesian Products

Definition 5.1. The Cartesian product of E and F is

$$E \times F = \{(x, y) \mid x \in E, y \in F\}.$$

Ordered pairs satisfy

$$(x, y) = (x', y') \iff x = x' \text{ and } y = y'.$$

Example 5.2. For $E = \{1, 2\}$ and $F = \{a, b\}$,

$$E \times F = \{(1, a), (1, b), (2, a), (2, b)\}.$$

Proposition 5.3. For arbitrary sets E and F ,

$$E \times \emptyset = \emptyset, \quad \emptyset \times F = \emptyset.$$

If E and F are finite, then

$$|E \times F| = |E| |F|.$$

Remark 5.4. Cartesian products are generally not commutative. However, $E \neq F$ does not always imply $E \times F \neq F \times E$: both products are empty if one of the factors is empty.

6 Partitions and Equivalence Relations

Definition 6.1 (Partition). A partition of E is a family $(A_i)_{i \in I}$ of nonempty subsets of E such that

$$E = \bigcup_{i \in I} A_i$$

and

$$i \neq j \Rightarrow A_i \cap A_j = \emptyset.$$

Every element of E belongs to exactly one block of a partition.

Example 6.2. Let $E = \{1, 2, 3\}$.

1. The family

$$\mathcal{P}_1 = \{\{1\}, \{2\}, \{3\}\}$$

forms a partition of E .

2. Another partition of E is

$$\mathcal{P}_2 = \{\{1, 2\}, \{3\}\}.$$

Definition 6.3 (Cover). A family $(B_i)_{i \in I}$ covers E if

$$E \subseteq \bigcup_{i \in I} B_i.$$

The sets in a cover may overlap and need not be subsets of E .

Example 6.4. Let $E = \{1, 2, 3\}$.

1. The family

$$\mathcal{C}_1 = \{\{1, -2\}, \{4, 5, 3\}, \{1\}\}$$

forms a covering of E .

2. The family

$$\mathcal{C}_2 = \{\{12\}, \{4, -1\}, \{3, 4, 7\}\}$$

also forms a covering of E .

3. Let $E = \mathbb{R}$. The family

$$\mathcal{B} = \{B_n =]-n, n[: n \in \mathbb{N}\}$$

forms a covering of E .

4. The family

$$\mathcal{A} = \{A_n =]n, n+1] : n \in \mathbb{Z}\}$$

forms both a covering and a partition of E .

Definition 6.5 (Equivalence relation). A relation $\sim$ on E is an equivalence relation if it is:

1. reflexive:

$$x \sim x \quad (x \in E);$$

2. symmetric:

$$x \sim y \Rightarrow y \sim x;$$

3. transitive:

$$x \sim y \text{ and } y \sim z \Rightarrow x \sim z.$$

Definition 6.6 (Equivalence class). The equivalence class of $x \in E$ is

$$[x] = \{y \in E \mid x \sim y\}.$$

The quotient set is

$$E/\sim = \{[x] \mid x \in E\}.$$

7 Functions

Definition 7.1. A function $f : E \rightarrow F$ assigns to every $x \in E$ exactly one element $f(x) \in F$.

The set E is its domain, and F is its codomain.

As a graph, f is a subset of $E \times F$ satisfying:

$$\forall x \in E, \quad \exists y \in F, \quad (x, y) \in f,$$

and

$$(x, y) \in f, (x, y') \in f \Rightarrow y = y'.$$

7.1 Injective, surjective, and bijective functions

Definition 7.2. A function $f : E \rightarrow F$ is:

1. injective if

$$f(x) = f(x') \Rightarrow x = x';$$

2. surjective if

$$\forall y \in F, \quad \exists x \in E, \quad f(x) = y;$$

3. bijective if it is both injective and surjective.

Example 7.3. The function

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = 2x + 1$$

is bijective.

The function

$$g : \mathbb{R} \rightarrow \mathbb{R}, \quad g(x) = x^2$$

is neither injective nor surjective.

However,

$$h : [0, \infty) \rightarrow [0, \infty), \quad h(x) = x^2$$

is bijective.

7.2 Direct images and inverse images

For $A \subseteq E$, its direct image is

$$f(A) = \{f(x) \mid x \in A\}.$$

For $B \subseteq F$, its inverse image is

$$f^{-1}(B) = \{x \in E \mid f(x) \in B\}.$$

An inverse image is defined even when f is not bijective.

Proposition 7.4. *For families $A_i \subseteq E$ and $B_i \subseteq F$,*

$$f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i),$$

$$f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i),$$

and, for nonempty I ,

$$f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i).$$

Also,

$$f^{-1}(F \setminus B) = E \setminus f^{-1}(B).$$

7.3 Images of intersections

For $A, B \subseteq E$,

$$f(A \cap B) \subseteq f(A) \cap f(B).$$

Equality need not hold. For example, if $f(x) = x^2$, $A = \{-1\}$, and $B = \{1\}$, then

$$f(A \cap B) = \emptyset, \quad f(A) \cap f(B) = \{1\}.$$

If f is injective, equality does hold:

$$f(A \cap B) = f(A) \cap f(B).$$

7.4 Useful image identities

For every $A \subseteq E$,

$$A \subseteq f^{-1}(f(A)).$$

If f is injective, then

$$f^{-1}(f(A)) = A.$$

For every $B \subseteq F$,

$$f(f^{-1}(B)) = B \cap f(E).$$

If f is surjective, this becomes

$$f(f^{-1}(B)) = B.$$

8 Composition and Inverse Functions

Definition 8.1 (Composition). Let $f : E \rightarrow F$ and $g : F \rightarrow G$. Their composition is

$$g \circ f : E \rightarrow G, \quad (g \circ f)(x) = g(f(x)).$$

The function f is applied first, followed by g .

Example 8.2. If $f(x) = 2x + 1$ and $g(x) = x^2$, then

$$(g \circ f)(x) = (2x + 1)^2,$$

whereas

$$(f \circ g)(x) = 2x^2 + 1.$$

Thus composition is not generally commutative.

Proposition 8.3. *Whenever the compositions are defined:*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Moreover,

$$f \circ \text{id}_E = f, \quad \text{id}_F \circ f = f,$$

where $\text{id}_E(x) = x$.

Proposition 8.4. 1. *A composition of injective functions is injective.*

2. *A composition of surjective functions is surjective.*

3. *A composition of bijective functions is bijective.*

Definition 8.5 (Inverse function). If $f : E \rightarrow F$ is bijective, its inverse is the function

$$f^{-1} : F \rightarrow E$$

defined by

$$f^{-1}(y) = x \iff f(x) = y.$$

It satisfies

$$f^{-1} \circ f = \text{id}_E, \quad f \circ f^{-1} = \text{id}_F.$$

Example 8.6. If

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = 3x - 2,$$

then

$$f^{-1}(y) = \frac{y + 2}{3}.$$

Proposition 8.7. *If $f : E \rightarrow F$ and $g : F \rightarrow G$ are bijective, then*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Remark 8.8. The inverse function f^{-1} requires bijectivity. The inverse image $f^{-1}(B)$ of a set does not.

9 Finite and Countable Sets

Definition 9.1 (Equipotence). Two sets A and B are equipotent if there exists a bijection between them. We write

$$A \sim B \quad \text{or} \quad |A| = |B|.$$

Definition 9.2 (Finite set). A set E is finite if it is empty or is in bijection with $\{1, \dots, n\}$ for some positive integer n . Its cardinality is then n .

Definition 9.3 (Countability). A set is countably infinite if it is in bijection with $\mathbb{N}$. It is at most countable if it is finite or countably infinite.

The cardinality of $\mathbb{N}$ is denoted by

$$\aleph_0.$$

Example 9.4. The function

$$f : \mathbb{N} \rightarrow \{0, 2, 4, \dots\}, \quad f(n) = 2n,$$

is bijective.

Thus an infinite set may have the same cardinality as a proper subset.

Example 9.5. The integers can be listed as

$$0, 1, -1, 2, -2, 3, -3, \dots$$

and are therefore countably infinite.

Remark 9.6. In the usual framework with the axiom of choice:

1. every subset of an at most countable set is at most countable;
2. a countable union of at most countable sets is at most countable;
3. every infinite set contains a countably infinite subset.

10 Comparing Cardinalities

We write

$$|A| \leq |B|$$

if there exists an injection from A into B .

We write

$$|A| < |B|$$

if there is such an injection but no bijection between A and B .

Theorem 10.1 (Cantor–Bernstein). *If there is an injection $A \rightarrow B$ and an injection $B \rightarrow A$, then there is a bijection $A \rightarrow B$.*

This theorem allows us to establish equal cardinalities without constructing a bijection directly.

11 Cantor's Theorem

Theorem 11.1. *For every set E ,*

$$|E| < |\mathcal{P}(E)|.$$

Proof. First, the function

$$i : E \rightarrow \mathcal{P}(E), \quad i(x) = \{x\},$$

is injective.

Now suppose that a surjection

$$f : E \rightarrow \mathcal{P}(E)$$

exists. Define

$$D = \{x \in E \mid x \notin f(x)\}.$$

Since $D \in \mathcal{P}(E)$ and f is surjective, there is $a \in E$ such that $f(a) = D$. Then

$$a \in D \iff a \notin f(a) \iff a \notin D,$$

a contradiction.

Hence no surjection from E onto $\mathcal{P}(E)$ exists. In particular, there is no bijection. $\square$

Corollary 11.2. *There is no largest cardinality. In particular,*

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots.$$

12 The Uncountability of the Real Numbers

Theorem 12.1. *The interval $(0, 1)$ is uncountable.*

Proof. Suppose that its elements can be listed as

$$x_1, x_2, x_3, \dots$$

Write their decimal expansions:

$$x_1 = 0.a_{11}a_{12}a_{13} \dots,$$

$$x_2 = 0.a_{21}a_{22}a_{23} \dots,$$

$$x_3 = 0.a_{31}a_{32}a_{33} \dots$$

For each number, choose the decimal expansion that is not eventually all 9's.

Define

$$y = 0.b_1b_2b_3 \dots$$

by

$$b_n = \begin{cases} 1, & a_{nn} \neq 1, \\ 2, & a_{nn} = 1. \end{cases}$$

Then $y \in (0, 1)$, but y differs from x_n in its n th decimal digit for every n .

Thus y does not occur in the supposedly complete list, which is a contradiction. $\square$

Corollary 12.2. *The real numbers are uncountable, and*

$$\aleph_0 < |\mathbb{R}|.$$

The cardinality of $\mathbb{R}$ is called the cardinality of the continuum and is denoted by

$$\mathfrak{c} = |\mathbb{R}|.$$

13 Orders and the Axiom of Choice

Definition 13.1 (Partial order). A relation $\leq$ on E is a partial order if it is:

1. reflexive: $x \leq x$;

2. antisymmetric:

$$x \leq y \text{ and } y \leq x \Rightarrow x = y;$$

3. transitive:

$$x \leq y \text{ and } y \leq z \Rightarrow x \leq z.$$

An order is total if every two elements are comparable.

Definition 13.2 (Chain). A chain in a partially ordered set is a subset on which the order is total.

Definition 13.3 (Upper bound). An element $u \in E$ is an upper bound of $A \subseteq E$ if

$$a \leq u \quad \text{for every } a \in A.$$

Definition 13.4 (Maximal element). An element $m \in E$ is maximal if

$$m \leq x \Rightarrow x = m.$$

A maximal element need not be a greatest element, since some elements may be incomparable with it.

13.1 The axiom of choice

For every indexed family $(E_i)_{i \in I}$ of nonempty sets, there exists a function

$$\varphi : I \rightarrow \bigcup_{i \in I} E_i$$

such that

$$\varphi(i) \in E_i \quad \text{for every } i \in I.$$

Such a function is called a choice function.

13.2 The well-ordering theorem

A well-order is a total order in which every nonempty subset has a least element.

The well-ordering theorem states that every set can be well-ordered.

13.3 Zorn's lemma

Lemma 13.5 (Zorn). *Let E be a nonempty partially ordered set. If every chain in E has an upper bound in E , then E has a maximal element.*

Over Zermelo–Fraenkel set theory, the axiom of choice, the well-ordering theorem, and Zorn's lemma are equivalent.

14 Exercises

14.1 Set operations

Exercise 1. Let

$$E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$
$$A = \{0, 2, 4, 6, 8\}, \quad B = \{1, 2, 3, 5, 8\}, \quad C = \{0, 1, 2, 3, 4\}.$$

1. Find $A \cup B$, $A \cap B$, $A \setminus B$, and $B \setminus A$.
2. Find the complements of A , B , and C in E .
3. Compute $A \Delta B$.
4. Verify

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Exercise 2. Prove:

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C),$$
$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C),$$
$$(A \setminus B) \setminus C = A \setminus (B \cup C),$$
$$A \cap (B \setminus C) = (A \cap B) \setminus C.$$

Exercise 3. Prove each true statement and give a counterexample to each false statement.

1. $A \cup B = A \cup C \Rightarrow B = C$.
2. $A \cap B = A \cap C \Rightarrow B = C$.
3. $A \setminus B = \emptyset \iff A \subseteq B$.
4. $A \Delta B = A \cup B \iff A \cap B = \emptyset$.
5. $A \cap B = A \cup B \iff A = B$.
6. If $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then $B = C$.

Exercise 4. 1. Prove that

$$A \Delta B = B \Delta A.$$

2. Prove associativity:

$$A \Delta (B \Delta C) = (A \Delta B) \Delta C.$$

3. Prove that

$$(A \Delta B) \Delta B = A.$$

4. Solve $X \Delta A = B$ for X .

5. Prove that

$$A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C).$$

6. Determine all pairs (A, B) satisfying

$$A \Delta B = A \cap B.$$

Exercise 5. Let $E = \{a, b, c\}$, where the three elements are distinct.

1. List all elements of $\mathcal{P}(E)$.
2. Find $|\mathcal{P}(E)|$ and $|\mathcal{P}(\mathcal{P}(E))|$.
3. Decide whether each statement is true:

$$\begin{aligned} \{a\} \subseteq E, \quad \{a\} \in \mathcal{P}(E), \quad \{\{a\}\} \subseteq \mathcal{P}(E), \\ \emptyset \in \mathcal{P}(E), \quad \{\emptyset\} \subseteq \mathcal{P}(E). \end{aligned}$$

Exercise 6. 1. Prove

$$A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B).$$

2. Prove

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B).$$

3. Prove

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B).$$

4. Give an example where the last inclusion is strict.

Exercise 7. 1. List $A \times B$ and $B \times A$ for

$$A = \{1, 2\}, \quad B = \{a, b, c\}.$$

2. Prove

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

3. Prove

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

4. If $A \neq \emptyset$, prove

$$A \times B = A \times C \Rightarrow B = C.$$

Exercise 8. On $\mathbb{Z}$, define

$$x \sim y \iff 4 \text{ divides } x - y.$$

1. Prove that $\sim$ is an equivalence relation.
2. Describe $[0]$, $[1]$, $[2]$, and $[3]$.
3. Find the classes of 7, -1 , and 12.
4. Describe $\mathbb{Z}/\sim$.

Exercise 9. For each relation on $\mathbb{R}$, determine whether it is reflexive, symmetric, and transitive:

$$xR_1y \iff x - y \in \mathbb{Z},$$

$$xR_2y \iff |x - y| \leq 1,$$

$$xR_3y \iff x^2 = y^2.$$

For each equivalence relation obtained, describe the classes of 0 and 1.

Exercise 10. Let

$$E = \{1, 2, 3, 4, 5, 6, 7\}$$

and

$$P = \{\{1, 3, 5\}, \{2, 4\}, \{6, 7\}\}.$$

Define xRy if x and y belong to the same block.

1. Verify that P is a partition.
2. Prove that R is an equivalence relation.
3. Determine all equivalence classes.
4. How many ordered pairs belong to the graph of R ?

Exercise 11. Determine whether each function is injective, surjective, or bijective:

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R}, & f(x) &= x^2, \\ g : \mathbb{R} &\rightarrow [0, \infty), & g(x) &= x^2, \\ h : [0, \infty) &\rightarrow [0, \infty), & h(x) &= x^2, \\ u : \mathbb{Z} &\rightarrow \mathbb{Z}, & u(n) &= 2n + 1. \end{aligned}$$

Find the inverse of every bijective function.

Exercise 12. Let

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = (x - 1)^2.$$

Find:

$$\begin{aligned} f([-1, 3]), & \quad f([2, 4]), \\ f^{-1}(\{0\}), & \quad f^{-1}(\{4\}), \\ f^{-1}([1, 4]), & \quad f^{-1}((-\infty, 0)). \end{aligned}$$

Exercise 13. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = x + 3, \quad g(x) = 2x^2.$$

1. Compute $g \circ f$ and $f \circ g$.
2. Are these functions equal?
3. Prove that f is bijective and find its inverse.

Exercise 14. Consider

$$h : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R} \setminus \{2\}, \quad h(x) = \frac{2x + 1}{x - 1}.$$

1. Verify that $h(x) \neq 2$ for every $x \neq 1$.
2. Solve $y = h(x)$ for x .
3. Prove that h is bijective.
4. Find h^{-1} .

Exercise 15. Let $f : E \rightarrow F$ and $g : F \rightarrow G$.

1. If $g \circ f$ is injective, prove that f is injective.
2. If $g \circ f$ is surjective, prove that g is surjective.
3. Give an example where $g \circ f$ is bijective, although f is not surjective and g is not injective.

Exercise 16. Prove that $f : E \rightarrow F$ is injective if and only if

$$f(A \cap B) = f(A) \cap f(B)$$

for every $A, B \subseteq E$.

Hint: For the reverse implication, use singleton sets.

Exercise 17. Prove:

$$f \text{ is injective} \iff \forall A \subseteq E, \quad f^{-1}(f(A)) = A,$$

and

$$f \text{ is surjective} \iff \forall B \subseteq F, \quad f(f^{-1}(B)) = B.$$

Exercise 18. Construct a bijection between:

1. $\mathbb{N}$ and $\{1, 2, 3, \dots\}$;
2. $\mathbb{N}$ and $\{0, 3, 6, 9, \dots\}$;
3. $\mathbb{Z}$ and the set of odd integers;
4. $(0, 1)$ and (a, b) , where $a < b$.

Verify injectivity and surjectivity in each case.

Exercise 19 (Countability of $\mathbb{N}^2$). For $s \in \mathbb{N}$, let

$$D_s = \{(p, q) \in \mathbb{N}^2 \mid p + q = s\}.$$

1. List D_0, D_1, D_2, D_3 .
2. Find $|D_s|$.
3. Explain how to enumerate $\mathbb{N}^2$ by successive diagonals.
4. Prove that

$$\pi(p, q) = \frac{(p+q)(p+q+1)}{2} + q$$

defines a bijection from $\mathbb{N}^2$ to $\mathbb{N}$.

Hint: For fixed $p + q = s$, determine the consecutive integers attained by $\pi(p, q)$.

Exercise 20. 1. Write every positive rational as p/q with $p, q \geq 1$.

2. Arrange these fractions by increasing values of $p + q$.
3. Explain how removing repetitions gives an enumeration of the positive rationals.
4. Deduce an enumeration of $\mathbb{Q}$.

Exercise 21 (Cantor's theorem). Let $f : E \rightarrow \mathcal{P}(E)$ and define

$$D = \{x \in E \mid x \notin f(x)\}.$$

1. Explain why $D \in \mathcal{P}(E)$.
2. Prove that $D \neq f(a)$ for every $a \in E$.
3. Conclude that f is not surjective.

Exercise 22. Let

$$S = \{(u_n)_{n \geq 1} \mid u_n \in \{0, 1\}\}.$$

Suppose its elements could be listed as

$$u^{(1)}, u^{(2)}, u^{(3)}, \dots$$

Define

$$v_n = 1 - u_n^{(n)}.$$

1. Prove that $v \in S$.
2. Show that $v \neq u^{(n)}$ for every n .
3. Conclude that S is uncountable.

Exercise 23. For $A \subseteq E$, define

$$\mathbf{1}_A : E \rightarrow \{0, 1\}, \quad \mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

1. Prove that

$$\mathbf{1}_A = \mathbf{1}_B \iff A = B.$$

2. Given $u : E \rightarrow \{0, 1\}$, find $A \subseteq E$ such that $u = \mathbf{1}_A$.
3. Deduce a bijection between $\mathcal{P}(E)$ and the set of functions from E to $\{0, 1\}$.
4. When $|E| = n$, recover $|\mathcal{P}(E)| = 2^n$.

Exercise 24 (Equivalence relation induced by a function). Let $f : E \rightarrow F$. Define

$$x \sim y \iff f(x) = f(y).$$

1. Prove that $\sim$ is an equivalence relation.
2. Show that

$$[x] = f^{-1}(\{f(x)\}).$$

3. Verify that the formula

$$\bar{f} : E/\sim \rightarrow f(E), \quad \bar{f}([x]) = f(x)$$

is well-defined.

4. Prove that $\bar{f}$ is injective.
5. Prove that $\bar{f}$ is surjective.
6. Conclude that $E/\sim$ and $f(E)$ are equipotent.
7. Apply this construction to

$$f : \mathbb{Z} \rightarrow \mathbb{N}, \quad f(n) = n^2.$$

Describe the equivalence classes and the image.