

Example: Hough Transform for Line Detection

1. Problem Statement

Apply the Hough transform to detect the line in the following binary image I of size 3×3 , in which the pixels with value 1 are the edge pixels (Table 1). We use the usual image coordinates: x is the column index, y is the row index, both starting at 0, the origin $(0,0)$ is the top-left pixel, and the y axis points downwards.

Table 1: The binary input image I .

	$x=0$	$x=1$	$x=2$
$y=0$	1	0	0
$y=1$	0	1	0
$y=2$	0	0	1

2. From Image Space (x, y) to Parameter Space (θ, ρ)

A line cannot be described in the form $y = mx + b$ when it is vertical ($m \rightarrow \infty$). The Hough transform therefore uses the **polar parameterization** of a line: ρ is the signed distance from the origin to the line, and θ is the angle between the x axis and the normal to the line (Figure 1a). A point (x, y) belongs to the line (ρ, θ) if and only if:

$$\rho = x \cos \theta + y \sin \theta \quad (1)$$

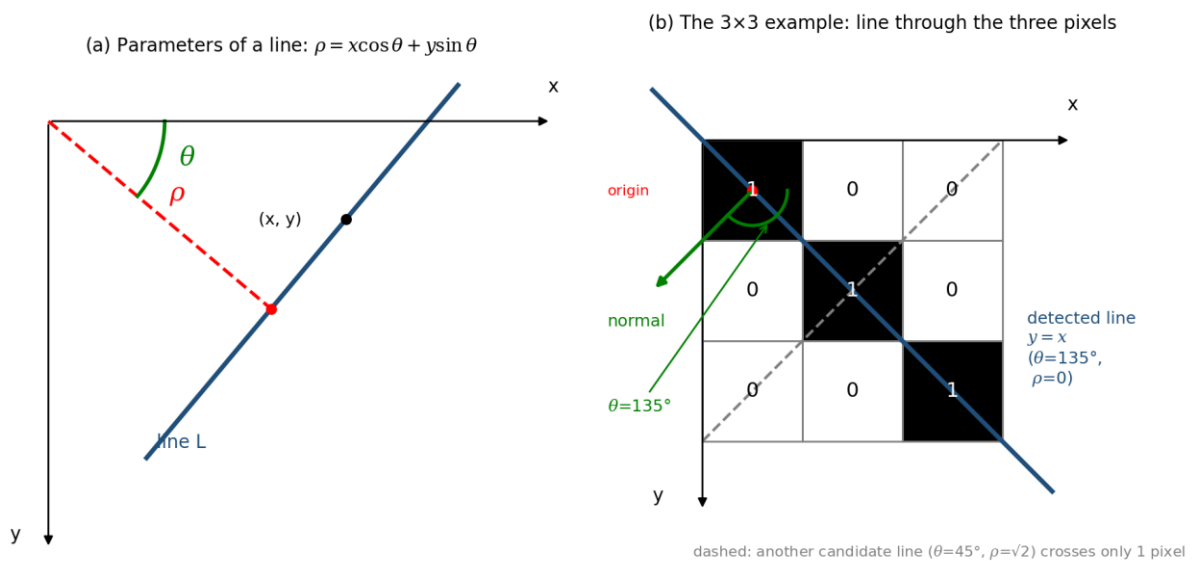


Figure 1: Parameters ρ and θ of a line (a) and the line detected in the 3×3 example (b).

This equation can be read in two ways, which is the key of the method:

- for a **fixed line** (ρ, θ) , it is the equation of a line in the image space (x, y) ;
- for a **fixed pixel** (x, y) , it is the equation of a **sinusoid** $\rho(\theta)$ in the parameter space (θ, ρ) : all the lines that pass through that pixel.

Consequently, **collinear pixels** give sinusoids that all intersect at a single point (θ_0, ρ_0) of the parameter space, and these parameters are those of the line that contains the pixels. The algorithm finds this intersection by **voting** in an accumulator $A(\rho, \theta)$. For an image of width W and height H , the parameters vary in the following ranges, where D is the diagonal of the image:

$$0^\circ \leq \theta < 180^\circ, \quad -D \leq \rho \leq D, \quad D = \sqrt{W^2 + H^2} \quad (2)$$

For this 3×3 image, $D = \sqrt{18} \approx 4.24$; in practice, only the values $|\rho| \leq 2\sqrt{2} \approx 2.83$ will occur.

3. Step 1: Extraction of the Edge Pixels

The image is already binary, so the edge pixels are the pixels with value 1. There are three of them (Table 2); they are located on the main diagonal.

Table 2: Edge pixels of the image.

Name	Coordinates (x, y)
P_1	(0,0)
P_2	(1,1)
P_3	(2,2)

4. Step 2: Quantization of the Parameter Space

The accumulator is a table of cells, each one corresponding to a pair (ρ, θ) . To keep the example small, we choose a coarse resolution:

- angle step $\Delta\theta = 45^\circ$, so $\theta \in \{0^\circ, 45^\circ, 90^\circ, 135^\circ\}$;
- distance step $\Delta\rho = 1$ pixel: each computed value of ρ is rounded to the nearest integer.

The accumulator $A(\rho, \theta)$ is initialized to zero in all the cells. For this image, no negative value of ρ appears (the three pixels satisfy $y = x$, so $\rho = 0$ at $\theta = 135^\circ$), and the rows $\rho < 0$ are therefore omitted.

5. Step 3: Voting, Pixel by Pixel

For each edge pixel and for each angle θ , we compute $\rho = x \cos \theta + y \sin \theta$ with the equation of Section 2, round it, and add 1 to the cell $A(\rho, \theta)$. The values of the trigonometric functions are:

$$\begin{aligned} \cos 0^\circ = 1, \sin 0^\circ = 0; \quad \cos 45^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2} \\ \approx 0.7071; \quad \cos 90^\circ = 0, \sin 90^\circ = 1; \quad \cos 135^\circ \\ = -0.7071, \sin 135^\circ = 0.7071 \end{aligned} \quad (3)$$

5.1. Pixel P1 = (0, 0)

Since $x = y = 0$, we have $\rho = 0$ for every angle: this pixel votes for the four cells of the row $\rho = 0$.

$$\rho = 0 \cdot \cos \theta + 0 \cdot \sin \theta = 0 \quad \text{for } \theta = 0^\circ, 45^\circ, 90^\circ, 135^\circ \quad (4)$$

Accumulator after P1

	$\theta=0^\circ$	$\theta=45^\circ$	$\theta=90^\circ$	$\theta=135^\circ$
$\rho=0$	1	1	1	1
$\rho=1$	0	0	0	0
$\rho=2$	0	0	0	0
$\rho=3$	0	0	0	0

5.2. Pixel P2 = (1, 1)

$$\theta = 0^\circ: \quad \rho = 1 \cdot 1 + 1 \cdot 0 = 1 \quad (5)$$

$$\theta = 45^\circ: \quad \rho = 1 \cdot 0.7071 + 1 \cdot 0.7071 = 1.414 \Rightarrow 1 \quad (6)$$

$$\theta = 90^\circ: \quad \rho = 1 \cdot 0 + 1 \cdot 1 = 1 \quad (7)$$

$$\theta = 135^\circ: \quad \rho = 1 \cdot (-0.7071) + 1 \cdot 0.7071 = 0 \quad (8)$$

The votes go to the cells $(\rho, \theta) = (1, 0^\circ), (1, 45^\circ), (1, 90^\circ)$ and $(0, 135^\circ)$.

Accumulator after P1 and P2

	$\theta=0^\circ$	$\theta=45^\circ$	$\theta=90^\circ$	$\theta=135^\circ$
$\rho=0$	1	1	1	2
$\rho=1$	1	1	1	0
$\rho=2$	0	0	0	0
$\rho=3$	0	0	0	0

5.3. Pixel P3 = (2, 2)

$$\theta = 0^\circ: \quad \rho = 2 \cdot 1 + 2 \cdot 0 = 2 \quad (9)$$

$$\theta = 45^\circ: \quad \rho = 2 \cdot 0.7071 + 2 \cdot 0.7071 = 2.828 \Rightarrow 3 \quad (10)$$

$$\theta = 90^\circ: \quad \rho = 2 \cdot 0 + 2 \cdot 1 = 2 \quad (11)$$

$$\theta = 135^\circ: \quad \rho = 2 \cdot (-0.7071) + 2 \cdot 0.7071 = 0 \quad (12)$$

The votes go to the cells $(2,0^\circ)$, $(3,45^\circ)$, $(2,90^\circ)$ and $(0,135^\circ)$. All the calculations are summarized in Table 3.

Table 3: Voting calculations: $\rho = x \cos \theta + y \sin \theta$ for each edge pixel and each angle (exact value $\rightarrow$ rounded value).

Pixel	$\theta = 0^\circ$	$\theta = 45^\circ$	$\theta = 90^\circ$	$\theta = 135^\circ$
$P_1 = (0,0)$	$0 \rightarrow 0$	$0 \rightarrow 0$	$0 \rightarrow 0$	$0 \rightarrow 0$
$P_2 = (1,1)$	$1 \rightarrow 1$	$1.414 \rightarrow 1$	$1 \rightarrow 1$	$0 \rightarrow 0$
$P_3 = (2,2)$	$2 \rightarrow 2$	$2.828 \rightarrow 3$	$2 \rightarrow 2$	$0 \rightarrow 0$

6. Step 4: Final Accumulator and Peak Detection

After the three pixels have voted, the accumulator is the one of Table 4. Each column sums to 3, because each pixel casts exactly one vote per angle.

Table 4: Final accumulator $A(\rho, \theta)$.

	$\theta=0^\circ$	$\theta=45^\circ$	$\theta=90^\circ$	$\theta=135^\circ$
$\rho=0$	1	1	1	3
$\rho=1$	1	1	1	0
$\rho=2$	1	0	1	0
$\rho=3$	0	1	0	0

All the cells contain at most 1 vote, except the cell $(\rho, \theta) = (0, 135^\circ)$, which contains **3 votes**: the three sinusoids of the pixels intersect there (Figure 2). With a vote threshold $T = 3$ (all the edge pixels), this cell is the only peak. The cells with a single vote correspond to lines that pass through only one of the pixels.

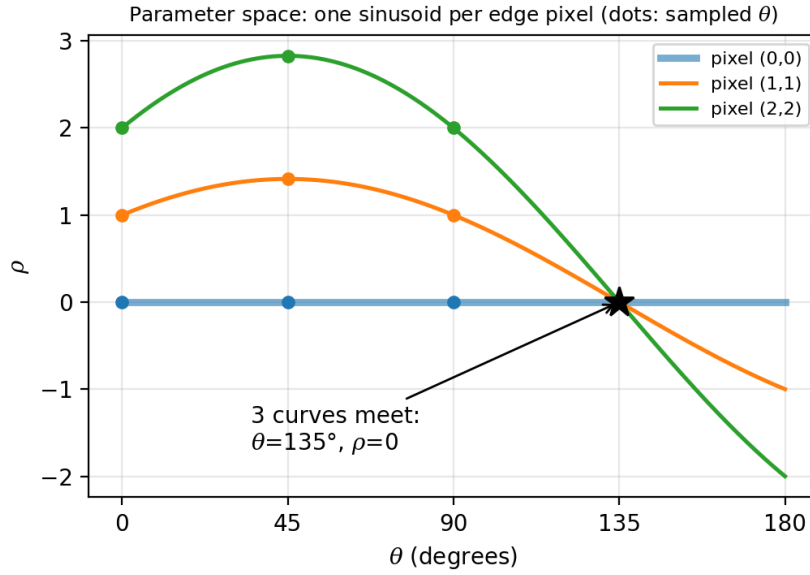


Figure 2: Parameter space: the sinusoids of the three edge pixels intersect at $(\theta = 135^\circ, \rho = 0)$.

7. Step 5: Back to the Image Space

The peak $(\rho, \theta) = (0, 135^\circ)$ is converted back to a line in the image space by substituting it into the polar equation:

$$x \cos 135^\circ + y \sin 135^\circ = 0 \Rightarrow -\frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} y = 0 \Rightarrow y = x \quad (13)$$

The detected line is therefore the main diagonal $y = x$ (slope 1, intercept 0). Its normal makes an angle of 135° with the x axis and it passes through the origin ($\rho = 0$), as shown in Figure 1b. Check: the three edge pixels satisfy the equation of the line, because for each of them $x \cos 135^\circ + y \sin 135^\circ = (y - x)\sqrt{2}/2 = 0$.

Remarks

- **Orientation convention:** with the origin at the top-left and the y axis pointing downwards (as here), the main diagonal has $\theta = 135^\circ$ and $\rho = 0$. If the y axis pointed upwards (origin at the bottom-left), the same pixels would lie on the line $x + y = 2$ and would give $\theta = 45^\circ$ and $\rho = 2/\sqrt{2} = \sqrt{2} \approx 1.41$.
- **Resolution:** the true orientation of the line (135°) is one of the sampled angles, so all three votes fall in the same cell. For a line with another orientation, the votes would be spread over neighbouring cells; a finer $\Delta\theta$ and $\Delta\rho$ gives more accurate parameters but needs a larger accumulator.
- **Noise and gaps:** the voting makes the method tolerant to missing pixels and to isolated noise pixels, which only add isolated single votes.

Table 5: Summary of the steps of the example.

Step	Operation	Result
1	Extract the edge pixels	$P_1(0,0), P_2(1,1), P_3(2,2)$
2	Quantize (ρ, θ)	$\Delta\theta = 45^\circ, \Delta\rho = 1$, accumulator initialized to 0
3	Voting: $\rho = x\cos\theta + y\sin\theta$	12 votes (3 pixels $\times$ 4 angles), see Table 3
4	Peak detection	$A(0, 135^\circ) = 3$
5	Back to the image space	Line $y = x$ ($\theta = 135^\circ, \rho = 0$)