

Chapter 1: Generalities

Table des matières

1	Introduction	2
2	Definitions	2
2.1	Different Types of Differential Equations	2
2.2	Linear Differential Equations	3
3	Some Techniques for Solving Differential Equations	3
3.1	Separable Differential Equations	3
3.2	Scalar Autonomous Equations	4
3.3	Linear Equations	4
3.4	Homogeneous Linear Equation	5
3.5	Nonhomogeneous Linear Equation	5
3.6	Duhamel's Formula	6
3.7	Bernoulli Equations	6
3.8	Riccati Equation	6
4	Reduction to First-Order Systems	7
5	Cauchy Problems(Initial Value Problem)	7
5.1	Local, Maximal, and Global Solutions	8
5.2	Examples	8
6	Cauchy–Lipschitz Theorem	9
6.1	Lipschitz Continuity	9
7	Dependence on Initial Conditions and Parameters	10
7.1	Gronwall's Lemma	10
7.2	Dependence on Initial Conditions	11
7.3	Dependence on Parameters	11
8	Linear Differential Systems	12
8.0.1	The case of a homogeneous system $\dot{x} = A(t)x$	13
8.0.2	The case of a non-homogeneous system $\dot{x} = A(t)x + B(t)$	13
8.1	Linear differential systems with constant coefficients	13
8.1.1	The homogeneous case	13
8.1.2	The non-homogeneous case	14

1 Introduction

The aim of this chapter is to study ordinary differential equations (of first order) of the form

$$x' = f(t, x)$$

where x is an unknown function of the real variable t and the mapping f is given as part of the problem. At each instant t , the value of $x(t)$ is called the *state of the system* at time t , and the space in which the unknown function x takes its values is called the *state space*.

Since mechanics is one of the main origins of the study of these equations, we will often refer to the variable t as the *time variable*, and to a solution $t \mapsto x(t)$ as the *trajectory of the system*. Throughout this course, the state space considered will be a subset of a finite-dimensional vector space.

Many questions arise concerning the above equation :

1. If an initial state of the system is given, that is, an initial time t_0 and the state x_0 of the system at time t_0 , does there exist one (or several) solution(s) of the equation satisfying the initial condition $x(t_0) = x_0$?
2. If such a solution exists, can it be calculated explicitly ?
3. If an explicit calculation is impossible, can the solution at least be described qualitatively ?
4. If f is a periodic function of time, do there exist time-periodic solutions of the equation ?
5. If x and y are two solutions of the equation associated with close initial data x_0 and y_0 , do the solutions x and y remain close to each other over time ?

2 Definitions

We begin with some essential definitions that will be used throughout this chapter.

2.1 Different Types of Differential Equations

Définition 2.1 (Ordinary Differential Equation). *An ordinary differential equation (ODE) is a relation involving the independent variable t , an unknown function $t \mapsto x(t)$, and its derivatives $x', x'', \dots, x^{(n)}$. It can be written in the form*

$$F(t, x(t), x'(t), \dots, x^{(n)}(t)) = 0, \tag{1}$$

where F depends on the independent variable t and the different derivatives of the unknown function.

Définition 2.2 (Normal Differential Equation). *An ordinary differential equation of order n is said to be in normal form if it can be written as*

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)). \tag{2}$$

Définition 2.3 (Autonomous Differential Equation). *A first-order differential equation is called autonomous if it has the form*

$$x' = f(x). \quad (3)$$

In other words, the independent variable t does not appear explicitly in the right-hand side.

For example,

$$x' = f(t, x) \quad (4)$$

is a first-order equation in normal form, whereas

$$x' = f(x) \quad (5)$$

is a first-order autonomous equation.

2.2 Linear Differential Equations

We now introduce a classification according to linearity.

Définition 2.4 (Linear Differential Equation). *An ordinary differential equation of order n is linear if it has the form*

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_1(t)x'(t) + a_0(t)x(t) = b(t), \quad (6)$$

where the coefficients $a_0, \dots, a_n$ and the function b depend at most on the independent variable t .

The unknown function and all its derivatives occur only to the first degree.

For example, a first-order linear differential equation can be written as

$$a(t)x'(t) + b(t)x(t) = c(t). \quad (7)$$

3 Some Techniques for Solving Differential Equations

Explicit solutions of ordinary differential equations are not always available. Nevertheless, several important classes of equations can be solved analytically.

In this section, we concentrate mainly on first-order scalar ordinary differential equations.

3.1 Separable Differential Equations

Consider a first-order equation in normal form

$$x' = f(t, x). \quad (8)$$

A particularly important case occurs when the right-hand side can be written as a product

$$f(t, x) = a(t)b(x). \quad (9)$$

The equation then becomes

$$x' = a(t)b(x). \quad (10)$$

Définition 3.1 (Separable Equation). *A first-order differential equation is called a separable equation if it can be written in the form*

$$b(x)x' = a(t), \quad (11)$$

where a and b are functions defined on suitable intervals.

Assume that A is a primitive of a and B is a primitive of b . Then

$$B'(x) = b(x) \quad (12)$$

and, along a solution,

$$\frac{d}{dt}B(x(t)) = b(x(t))x'(t) = a(t). \quad (13)$$

Therefore,

$$B(x(t)) = A(t) + C, \quad C \in \mathbb{R}. \quad (14)$$

This relation gives the implicit form of the solution.

3.2 Scalar Autonomous Equations

We now consider scalar autonomous equations of the form

$$x' = f(x). \quad (15)$$

If a is a root of f , that is,

$$f(a) = 0,$$

then the constant function

$$x(t) = a \quad (16)$$

is a solution.

Such a solution is called an equilibrium solution.

3.3 Linear Equations

We continue with first-order ordinary differential equations.

Définition 3.2 (First-Order Linear ODE). *A first-order differential equation is linear if it can be written in the form*

$$a(t)x'(t) + b(t)x(t) = c(t), \quad (17)$$

where a and b are continuous on an interval I and $a(t) \neq 0$ on I .

Dividing by $a(t)$ gives the standard form

$$x'(t) + p(t)x(t) = q(t). \quad (18)$$

3.4 Homogeneous Linear Equation

The homogeneous equation is

$$a(t)x'(t) + b(t)x(t) = 0. \quad (19)$$

Assuming $a(t) \neq 0$, it becomes

$$x'(t) + p(t)x(t) = 0. \quad (20)$$

This is a separable equation :

$$\frac{x'}{x} = -p(t), \quad (21)$$

for a nonzero solution.

Integrating gives

$$\ln |x(t)| = - \int p(t) dt + C, \quad (22)$$

and consequently

$$x(t) = C \exp \left(- \int p(t) dt \right). \quad (23)$$

The zero function

$$x(t) = 0$$

is also a solution and is called the trivial solution.

3.5 Nonhomogeneous Linear Equation

Consider

$$a(t)x'(t) + b(t)x(t) = c(t). \quad (24)$$

In standard form,

$$x'(t) + p(t)x(t) = q(t). \quad (25)$$

Let

$$P(t) = \int p(t) dt. \quad (26)$$

Multiplying the equation by the integrating factor

$$\mu(t) = e^{P(t)} \quad (27)$$

gives

$$\mu(t)x'(t) + \mu(t)p(t)x(t) = \mu(t)q(t). \quad (28)$$

Since

$$\mu'(t) = p(t)\mu(t),$$

we obtain

$$\frac{d}{dt} (\mu(t)x(t)) = \mu(t)q(t). \quad (29)$$

Hence

$$\mu(t)x(t) = C + \int \mu(t)q(t) dt, \quad (30)$$

and therefore

$$x(t) = e^{-P(t)} \left[C + \int e^{P(t)} q(t) dt \right]. \quad (31)$$

This method is frequently called the *variation of constants* method.

3.6 Duhamel's Formula

For the particular case

$$x'(t) + ax(t) = f(t), \quad (32)$$

where a is a constant, and given the initial condition

$$x(t_0) = x_0,$$

the solution is

$$x(t) = e^{-a(t-t_0)}x_0 + \int_{t_0}^t e^{-a(t-s)}f(s) ds. \quad (33)$$

This representation is known as *Duhamel's formula*.

3.7 Bernoulli Equations

Définition 3.3 (Bernoulli Equation). *A Bernoulli equation is a nonlinear first-order scalar differential equation of the form*

$$x' + P(t)x + Q(t)x^r = 0, \quad (34)$$

where P and Q are continuous functions and r is a real number.

The cases $r = 0$ and $r = 1$ reduce to linear equations.

For $x \neq 0$, divide by x^r :

$$x^{-r}x' + P(t)x^{1-r} + Q(t) = 0. \quad (35)$$

Introduce the transformation

$$u = x^{1-r}. \quad (36)$$

Then

$$u' = (1-r)x^{-r}x'. \quad (37)$$

Consequently, the Bernoulli equation becomes the linear equation

$$u' + (1-r)P(t)u + (1-r)Q(t) = 0. \quad (38)$$

Thus, the Bernoulli equation can be reduced to a first-order linear differential equation.

The solution $x = 0$ must be considered separately whenever it is admissible.

3.8 Riccati Equation

The particular equation

$$x' = a(t)x^2 + b(t)x + c(t) \quad (39)$$

is called a *Riccati equation*.

4 Reduction to First-Order Systems

Before solving differential equations of arbitrary order, it is useful to observe that an equation of order n can be transformed into a first-order system by introducing additional variables.

Consider an ordinary differential equation of order n :

$$F(t, x, x', \dots, x^{(n)}) = 0. \quad (40)$$

Introduce the new variables

$$x_1 = x, \quad x_2 = x', \quad \dots, \quad x_n = x^{(n-1)}. \quad (41)$$

Then

$$x'_1 = x_2, \quad x'_2 = x_3, \quad \dots, \quad x'_{n-1} = x_n.$$

If the original equation can be written in normal form,

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}),$$

we obtain the first-order system

$$\begin{cases} x'_1 = x_2, \\ x'_2 = x_3, \\ \vdots \\ x'_{n-1} = x_n, \\ x'_n = f(t, x_1, x_2, \dots, x_n). \end{cases} \quad (42)$$

Thus, a higher-order differential equation can be transformed into a first-order system. This reduction is fundamental because most of the general theory is formulated for first-order systems.

5 Cauchy Problems(Initial Value Problem)

Let $I \subset \mathbb{R}$ be an open interval with nonempty interior, and let $t_0 \in I$. Let E be a Banach space and let D be an open, connected subset of E . Consider a mapping

$$f : D \times I \rightarrow E \quad (43)$$

and a point $y_0 \in D$.

Définition 5.1 (Cauchy Problem). *A Cauchy problem consists in finding an interval J , with $t_0 \in J \subseteq I$, and a mapping $y : J \rightarrow D$ such that*

$$\begin{cases} y'(t) = f(t, y(t)), & t \in J, \\ y(t_0) = y_0. \end{cases} \quad (44)$$

In most applications, we take $E = \mathbb{R}^d$, $d \in \mathbb{N}$, and assume that f is at least continuous. The differential formulation is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds, \quad t \in J. \quad (45)$$

5.1 Local, Maximal, and Global Solutions

Définition 5.2. *The following notions are used for solutions of the Cauchy problem.*

1. A pair (J, y) is called a *local solution* if $t_0 \in J \subseteq I$, $y \in C^1(J)$, J is a neighborhood of t_0 in I , and the Cauchy problem is satisfied for every $t \in J$.
2. Let (J_1, y_1) and (J_2, y_2) be two local solutions. We say that (J_1, y_1) *extends* (J_2, y_2) if

$$J_2 \subseteq J_1, \quad y_1|_{J_2} = y_2. \quad (46)$$

3. A local solution (J, y) is called a *maximal solution* if it has no proper extension which is also a solution.
4. A local solution (J, y) is called a *global solution* if $J = I$.

Remarque 5.1. *Every global solution is a maximal solution.*

If two solutions are defined on adjacent intervals and agree at their common point, they can be combined. More precisely, let $t_1 < t_0 < t_2$ and $t_3 < t_2 < t_4$, and let (J_1, y_1) and (J_2, y_2) be local solutions such that $[t_1, t_2] \subseteq J_1$ and $[t_3, t_4] \subseteq J_2$, with

$$y_1'(t) = f(t, y_1(t)), \quad y_1(t_0) = y_0, \quad (47)$$

and

$$y_2'(t) = f(t, y_2(t)), \quad y_2(t_2) = y_1(t_2). \quad (48)$$

Then the function y defined on $[t_1, t_4]$ by

$$y(t) = \begin{cases} y_1(t), & t \in [t_1, t_2], \\ y_2(t), & t \in [t_2, t_4], \end{cases} \quad (49)$$

is again a local solution extending the first piece.

If f is of class C^k on $I \times D$, then every local solution (J, y) is of class C^{k+1} on J .

5.2 Examples

Exemple 5.1. *Consider*

$$\begin{cases} \dot{y} = -2ty^2, \\ y(0) = 1, \end{cases} \quad I = \mathbb{R}. \quad (50)$$

The problem admits a unique global solution

$$y(t) = \frac{1}{1+t^2}. \quad (51)$$

Exemple 5.2. *Consider*

$$\begin{cases} \dot{y} = 2ty^2, \\ y(0) = 1, \end{cases} \quad I = \mathbb{R}. \quad (52)$$

The problem admits a unique maximal solution on $(-1, 1)$, namely

$$y(t) = \frac{1}{1-t^2}. \quad (53)$$

Exemple 5.3. *Consider*

$$\begin{cases} \dot{y} = -y^2, \\ y(0) = 1. \end{cases} \quad (54)$$

With $I = \mathbb{R}_+$, the problem admits the global solution

$$y(t) = \frac{1}{1+t}. \quad (55)$$

With $I = \mathbb{R}$, the corresponding maximal solution is

$$(J, y) = \left((-1, +\infty), \frac{1}{1+t} \right), \quad (56)$$

which is not global on $\mathbb{R}$.

Exemple 5.4. *Consider*

$$\begin{cases} \dot{y} = y^2, \\ y(0) = 1, \end{cases} \quad I = \mathbb{R}. \quad (57)$$

The problem admits the maximal solution

$$(J, y) = \left((-\infty, 1), \frac{1}{1-t} \right). \quad (58)$$

6 Cauchy–Lipschitz Theorem

In this section, we present the fundamental local existence and uniqueness theorem for the autonomous nonlinear system

$$\dot{x} = f(x). \quad (59)$$

The proof is based on successive approximations. The regularity assumptions of the vector field also play a fundamental role in studying the dependence of solutions on initial conditions and parameters.

6.1 Lipschitz Continuity

Définition 6.1. *Let E be an open subset of $\mathbb{R}^n$. A function $f : E \rightarrow \mathbb{R}^n$ is said to be Lipschitz continuous on E if there exists a constant $K \geq 0$ such that*

$$\|f(x) - f(y)\| \leq K\|x - y\| \quad \text{for all } x, y \in E. \quad (60)$$

Définition 6.2. *A function $f : E \rightarrow \mathbb{R}^n$ is said to be locally Lipschitz continuous on E if, for every $x_0 \in E$, there exist $\varepsilon > 0$ and $K_0 \geq 0$ such that $N_\varepsilon(x_0) \subset E$ and*

$$\|f(x) - f(y)\| \leq K_0\|x - y\| \quad \text{for all } x, y \in N_\varepsilon(x_0), \quad (61)$$

where

$$N_\varepsilon(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < \varepsilon\}. \quad (62)$$

Let E be an open subset of $\mathbb{R}^n$ and let $f : E \rightarrow \mathbb{R}^n$. If $f \in C^1(E)$, then f is locally Lipschitz continuous on E .

Démonstration. Let $x_0 \in E$. Since E is open, there exists $r > 0$ such that $\overline{B(x_0, r)} \subset E$, where $B(x_0, r)$ denotes the open ball centered at x_0 with radius r .

Since Df is continuous and $\overline{B(x_0, r)}$ is compact, there exists $K \geq 0$ such that

$$\|Df(x)\| \leq K \quad \text{for all } x \in \overline{B(x_0, r)}. \quad (63)$$

For any $x, y \in B(x_0, r)$, the mean value estimate gives

$$\|f(x) - f(y)\| \leq K\|x - y\|. \quad (64)$$

Therefore, f is locally Lipschitz continuous on E . $\square$

Théorème 6.1 (Cauchy–Lipschitz Theorem). *Let E be an open subset of $\mathbb{R}^n$, let $x_0 \in E$, and suppose that $f \in C^1(E)$. Then there exists $a > 0$ such that the initial value problem*

$$\begin{cases} \dot{x}(t) = f(x(t)), \\ x(0) = x_0, \end{cases} \quad (65)$$

admits a unique solution $x \in C^1([-a, a], E)$.

Remarque 6.1. *The theorem guarantees local existence and uniqueness. It does not, in general, guarantee that the solution exists for all $t \in \mathbb{R}$.*

7 Dependence on Initial Conditions and Parameters

We now study the dependence of solutions of an initial value problem on their initial conditions and on parameters. Consider

$$\begin{cases} \dot{x}(t) = f(x(t), \mu), \\ x(0) = y, \end{cases} \quad (66)$$

where $x(t) \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and $\mu \in \mathbb{R}^m$ is a parameter.

7.1 Gronwall's Lemma

[Gronwall's Lemma] Let $g : [0, a] \rightarrow [0, \infty)$ be a continuous function satisfying

$$g(t) \leq C + K \int_0^t g(s) ds, \quad t \in [0, a], \quad (67)$$

where $C \geq 0$ and $K \geq 0$ are constants. Then

$$g(t) \leq Ce^{Kt}, \quad t \in [0, a]. \quad (68)$$

Démonstration. Define

$$G(t) = C + K \int_0^t g(s) ds. \quad (69)$$

Then $g(t) \leq G(t)$ and $G(0) = C$. Moreover,

$$G'(t) = Kg(t) \leq KG(t). \quad (70)$$

If $C > 0$, then $G(t) > 0$, and hence

$$\frac{G'(t)}{G(t)} \leq K. \quad (71)$$

Integrating from 0 to t yields

$$\ln G(t) - \ln C \leq Kt. \quad (72)$$

Consequently,

$$G(t) \leq Ce^{Kt}. \quad (73)$$

Since $g(t) \leq G(t)$, we conclude that

$$g(t) \leq Ce^{Kt}. \quad (74)$$

If $C = 0$, the same conclusion follows by applying the preceding argument with any $\varepsilon > 0$ in place of C and then letting $\varepsilon \rightarrow 0^+$. $\square$

7.2 Dependence on Initial Conditions

Théorème 7.1 (Dependence on Initial Conditions). *Let E be an open subset of $\mathbb{R}^n$, let $x_0 \in E$, and suppose that $f \in C^1(E)$. Then there exist $a > 0$ and $\delta > 0$ such that, for every $y \in N_\delta(x_0)$, the initial value problem*

$$\begin{cases} \dot{x}(t) = f(x(t)), \\ x(0) = y, \end{cases} \quad (75)$$

admits a unique solution $u(t, y)$ for all $t \in [-a, a]$.

Moreover, the solution map

$$u : [-a, a] \times N_\delta(x_0) \rightarrow E, \quad (t, y) \mapsto u(t, y), \quad (76)$$

is of class C^1 on the interior of its domain. In particular, the solution depends continuously differentiably on the initial condition.

Remarque 7.1. *The C^1 regularity of the solution map follows from the assumption $f \in C^1(E)$. Higher regularity, such as C^2 or C^∞ , requires corresponding higher regularity assumptions on f .*

7.3 Dependence on Parameters

Théorème 7.2 (Dependence on Parameters). *Let E be an open subset of $\mathbb{R}^{n+m}$ containing (x_0, μ_0) , and let $f : E \rightarrow \mathbb{R}^n$ be of class $C^1(E)$. Then there exist $a > 0$, $\delta_x > 0$, and $\delta_\mu > 0$ such that, for every*

$$y \in N_{\delta_x}(x_0), \quad \mu \in N_{\delta_\mu}(\mu_0),$$

the initial value problem

$$\begin{cases} \dot{x}(t) = f(x(t), \mu), \\ x(0) = y, \end{cases} \quad (77)$$

admits a unique solution $u(t, y, \mu)$ for all $t \in [-a, a]$. Moreover, the solution map

$$(t, y, \mu) \mapsto u(t, y, \mu) \quad (78)$$

is of class C^1 on the interior of the corresponding product neighborhood.

Démonstration. Introduce the augmented variable

$$z = (x, \mu) \in \mathbb{R}^{n+m}$$

and define the vector field

$$F(x, \mu) = \begin{pmatrix} f(x, \mu) \\ 0 \end{pmatrix}. \quad (79)$$

Consider the augmented autonomous system

$$\begin{cases} \dot{x}(t) = f(x(t), \mu(t)), \\ \dot{\mu}(t) = 0. \end{cases} \quad (80)$$

Since $f \in C^1(E)$, the vector field F is also of class C^1 . The Cauchy–Lipschitz theorem and the theorem on dependence on initial conditions therefore apply to this augmented system.

Because $\dot{\mu}(t) = 0$, we have $\mu(t) = \mu(0) = \mu$ for all t in the interval of existence. Thus the first component satisfies

$$\dot{x}(t) = f(x(t), \mu), \quad x(0) = y.$$

The claimed local existence, uniqueness, and C^1 dependence on (t, y, μ) follow. $\square$

8 Linear Differential Systems

Many natural phenomena can be modelled, as a first approximation, by linear differential systems. Moreover, constant-coefficient linear systems can be solved completely. This gives such systems great practical importance.

A first-order linear differential system in $\mathbb{R}^n$ is an equation of the form

$$\dot{x} = A(t)x + B(t), \quad (2.1)$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \in \mathbb{R}^n$$

is the unknown function, $A(t) = (a_{ij}(t))_{1 \leq i, j \leq n} \in M_n(\mathbb{R})$, and

$$B(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix} \in \mathbb{R}^n$$

are given continuous functions.

Theorem 2.1. If the matrix-valued function A and the vector-valued function B are continuous on an interval I , then the initial-value problem

$$\begin{cases} \dot{x} = A(t)x + B(t), \\ x(t_0) = x_0, \end{cases} \quad (2.2)$$

where $t_0 \in I$ and x_0 is a constant vector, has a unique solution on the whole interval I .

For the proof, it is enough to observe that $F(t, x) = A(t)x + B(t)$ is continuous on I and is Lipschitz with constant $K = \|A\| = \sup_{|x|=1} |Ax|$. To solve the non-homogeneous system (2.2), we first need to solve the associated homogeneous system.

8.0.1 The case of a homogeneous system $\dot{x} = A(t)x$

Consider the homogeneous system associated with (2.2) :

$$\begin{cases} \dot{x} = A(t)x, \\ x(t_0) = x_0. \end{cases} \quad (2.3)$$

Let S be the set of maximal solutions. For any $x, y \in S$ and any $\lambda_1, \lambda_2 \in \mathbb{R}$, we have $\lambda_1 x + \lambda_2 y \in S$. Therefore, S is a vector subspace.

Corollary 2.2. The set S of maximal solutions is a vector space of dimension n over $\mathbb{R}$.

8.0.2 The case of a non-homogeneous system $\dot{x} = A(t)x + B(t)$

Return to the more general system (2.2). There exists at least one maximal solution y . If x is any solution, then $z = x - y$ satisfies the homogeneous equation (2.3), and conversely. Consequently, the set of maximal solutions is

$$y + S = \{y + z; z \in S\},$$

where S is the set of maximal solutions of the associated homogeneous equation. The set $y + S$ of maximal solutions of (2.3) is a translate of S and is therefore an affine space of dimension n over $\mathbb{R}$, with S as its direction vector space.

8.1 Linear differential systems with constant coefficients

8.1.1 The homogeneous case

Consider the homogeneous system with constant coefficients

$$\begin{cases} \dot{x} = Ax, \\ x(0) = x_0. \end{cases} \quad (2.4)$$

Using Picard iterations, we obtain

$$\begin{aligned} x_0(t) &= x_0, \\ x_1(t) &= x_0 + \int_0^t Ax_0 ds = x_0 + tAx_0, \end{aligned}$$

and

$$x_2(t) = x_0 + \int_0^t Ax_1(s) ds = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0. \quad (2.5)$$

By induction,

$$x_m(t) = \sum_{i=0}^m \frac{t^i}{i!} A^i x_0.$$

Passing to the limit gives

$$\lim_{m \rightarrow +\infty} x_m(t) = \sum_{i=0}^{+\infty} \frac{t^i}{i!} A^i x_0.$$

In the one-dimensional case ($n = 1$), this series is simply the exponential function. We therefore write

$$x(t) = \exp(tA)x_0,$$

and define the matrix exponential by

$$\exp(A) = \sum_{i=0}^{+\infty} \frac{1}{i!} A^i.$$

This matrix was presented in detail at the undergraduate level.

Lemma 2.3. Let A be a square matrix. Then

$$\frac{d}{dt} e^{At} = A e^{At}.$$

Proof. Since A commutes with itself, we have

$$\frac{d}{dt} e^{At} = \lim_{h \rightarrow 0} \frac{e^{A(t+h)} - e^{At}}{h} = e^{At} \lim_{h \rightarrow 0} \frac{e^{Ah} - I}{h} = A e^{At}.$$

The interchange of limits in the corresponding series is justified by its uniform convergence.

Theorem 2.4. Let A be a square matrix. For a given $x_0 \in \mathbb{R}^n$, the initial-value problem (2.4) has a unique solution given by

$$x(t) = e^{At} x_0.$$

Proof. If $x(t) = e^{At} x_0$, Lemma 2.3 gives $\dot{x}(t) = A e^{At} x_0$ for all $t \in \mathbb{R}$, and $x(0) = I x_0 = x_0$. Thus $x(t) = e^{At} x_0$ is a solution of (2.4).

Conversely, suppose that $x(t)$ is a solution of (2.4) and set $y(t) = e^{-At} x(t)$. By Lemma 2.3,

$$\dot{y}(t) = -A e^{-At} x(t) + e^{-At} \dot{x}(t) = -A e^{-At} x(t) + e^{-At} A x(t) = 0,$$

because A and e^{-At} commute. Hence $y(t)$ is constant, so $y(t) = y(0) = x(0) = x_0$. Therefore every solution of (2.4) is

$$x(t) = e^{At} y(t) = e^{At} x_0.$$

Example 2.1. Solve the initial-value problem

$$\begin{cases} \dot{x} = Ax, \\ x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{cases} \quad A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}.$$

Recall the matrix-exponential formulas

$$e^{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}} = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}, \quad e^{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}} = e^a \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

8.1.2 The non-homogeneous case

Consider the non-homogeneous constant-coefficient system

$$\begin{cases} \dot{x} = Ax + B(t), \\ x(0) = x_0, \end{cases} \tag{2.8}$$

where A is an $n \times n$ matrix and $B(t)$ is a continuous vector-valued function.

Definition 2.5. The matrix-valued function $\Phi(t)$, which is a non-singular $n \times n$ matrix and satisfies

$$\Phi'(t) = A\Phi(t) \quad \text{for all } t \in \mathbb{R},$$

is called a *fundamental matrix of solutions* of the associated homogeneous system

$$\dot{x} = Ax. \quad (2.9)$$

Remark 2.6. The function $\Phi(t) = e^{At}$ is a fundamental matrix of (2.9) satisfying $\Phi(0) = I$. Moreover, every fundamental matrix $\Phi(t)$ of (2.9) is of the form

$$\Phi(t) = e^{At}C,$$

where C is a non-singular matrix.

Theorem 2.7. If $\Phi(t)$ is a fundamental matrix of (2.9), then the unique solution of the non-homogeneous linear system (2.8) is

$$x(t) = \Phi(t)\Phi^{-1}(0)x_0 + \int_0^t \Phi(t)\Phi^{-1}(s)B(s) ds. \quad (2.10)$$

Proof. Differentiating (2.10), and using $\Phi'(t) = A\Phi(t)$ and $\Phi(t)\Phi^{-1}(t) = I$, gives

$$\dot{x}(t) = Ax(t) + B(t),$$

for all $t \in \mathbb{R}$.

Remark 2.8. With $\Phi(t) = e^{At}$, the solution of (2.8) is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}B(s) ds.$$

Example. Solve the forced harmonic oscillator

$$\ddot{x} + x = f(t). \quad (2.11)$$

This equation can be written as the system

$$\begin{cases} \dot{x}_1 = -x_2, \\ \dot{x}_2 = x_1 + f(t), \end{cases} \quad (2.12)$$

which has the matrix form $\dot{x} = Ax + B(t)$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

We have

$$e^{At} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad e^{-At} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Therefore, for the initial condition $x(0) = x_0$,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}B(s) ds = e^{At}x_0 + \int_0^t \begin{pmatrix} f(s) \sin(s-t) \\ f(s) \cos(t-s) \end{pmatrix} ds. \quad (2.13)$$

Thus the solution of (2.11) is

$$x(t) = x(0) \cos t - \dot{x}(0) \sin t - \int_0^t f(s) \sin(s-t) ds.$$