

Chapter 1

The C^∞ functions with compact support in an open set $\Omega \subset \mathbb{R}^n$

1.1 Space of test functions.

1.1.1 Support of function

Let Ω be a non-empty open subset of $\mathbb{R}^n$ ($n \geq 1$) and let $f : \Omega \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) be a continuous function.

Recall (Borel–Lebesgue theorem): A subset of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.

Definition of the support

Definition 1 *The support of a function f , denoted $\text{supp}(f)$, is the closure in Ω of the set of points where f does not vanish (i.e., f is nonzero):*

$$\text{supp}(f) = \overline{\{x \in \Omega \mid f(x) \neq 0\}}.$$

Proposition 2 *The support of f is a closed subset of $\mathbb{R}^n$. Moreover:*

$$x \notin \text{supp}(f) \iff \exists V \text{ an open neighborhood of } x \text{ such that } f = 0 \text{ on } V \cap \Omega.$$

In other words, $\text{supp}(f)$ is the smallest closed set outside of which f is zero.

Example 3 *In $\Omega = \mathbb{R}$, the function*

$$f(x) = \begin{cases} \exp\left(\frac{1}{x^2-1}\right), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

has support

$$\begin{aligned} \text{supp}(f) &= \overline{\{x \in \Omega : f(x) \neq 0\}} = \overline{\{x \in \Omega : |x| < 1\}} \\ &= \{x \in \Omega : |x| \leq 1\} = [-1, 1]. \end{aligned}$$

Example 4 In $\Omega = \mathbb{R}^n$. The function

$$f(x) = \begin{cases} \exp\left(\frac{1}{\|x\|^2 - 1}\right), & \|x\| < 1 \\ 0, & \|x\| \geq 1 \end{cases}$$

has support

$$\begin{aligned} \text{supp}(f) &= \overline{\{x \in \Omega : f(x) \neq 0\}} = \overline{\{x \in \Omega : \|x\| < 1\}} \\ &= \{x \in \Omega : \|x\| \leq 1\} = \overline{B}(0, 1) \end{aligned}$$

the closed ball centered at 0 with radius 1.

Functions with compact support in Ω

Definition 5 We say that $f : \Omega \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) has compact support in Ω if:

1. $\text{supp}(f)$ is compact (hence closed and bounded);
2. $\text{supp}(f) \subset \Omega$.

1.1.2 The space $C_c^\infty(\Omega)$

Function of class C^∞ on Ω

Definition 6 A function $\varphi : \Omega \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) is said to be of class C^∞ (or infinitely differentiable) on Ω if it has continuous partial derivatives of all orders.

Example 7 The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right), & \text{if } x > 0, \\ 0, & \text{if } x \leq 0 \end{cases}$$

is infinitely differentiable functions (i.e., of class C^∞ on $\mathbb{R}$).

We distinguish three cases:

Case 1: If $x < 0$, all derivatives of f vanish.

Case 2: If $x = 0$, the left-hand derivative of f vanishes.

Case 3: If $x > 0$, we have

$$f'(x) = \frac{2}{x^3} \exp\left(-\frac{1}{x^2}\right), \quad f''(x) = \frac{4 - 6x^2}{x^6} \exp\left(-\frac{1}{x^2}\right), \dots, \quad f^{(k)}(x) = \frac{P(x)}{x^{3k}} \exp\left(-\frac{1}{x^2}\right),$$

where $P(x)$ is a polynomial.

Consequently,

$$\begin{aligned} \lim_{x \rightarrow 0^+} f^{(k)}(x) &= \lim_{x \rightarrow 0^+} \left(\frac{P(x)}{x^{3k}} \exp\left(-\frac{1}{x^2}\right) \right) \\ &= P(0) \lim_{x \rightarrow 0^+} \left(\frac{1}{x^{3k}} \exp\left(-\frac{1}{x^2}\right) \right) = P(0) \lim_{u \rightarrow \infty} \left(\frac{1}{e^u} u^{\frac{3k}{2}} \right) = 0, \end{aligned}$$

it suffices to apply L'Hôpital's rule several times.

Finally, if $x = 0$, the right-hand derivatives of f vanish:

$$f^{(k)}(0) = 0, \quad \forall k \in \mathbb{N}.$$

Indeed, let us proceed by induction on k .

For $k = 0$, it is obvious. Suppose that

$$f^{(k)}(0) = 0$$

and let us show that

$$f^{(k+1)}(0) = 0.$$

We have

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x \rightarrow 0} \left(\frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} \right) = \lim_{x \rightarrow 0} \frac{f^{(k)}(x)}{x} \\ &= \lim_{x \rightarrow 0} P(x) \frac{\exp\left(-\frac{1}{x^2}\right)}{x^{3k+1}} = P(0) \lim_{x \rightarrow 0} \left(\frac{\exp\left(-\frac{1}{x^2}\right)}{x^{3k+1}} \right) \\ &= 0. \end{aligned}$$

Thus f is infinitely differentiable.

Space of test functions

Definition 8 We denote by $C_c^\infty(\Omega)$ the set of functions $\varphi : \Omega \rightarrow \mathbb{R}$ (or $\mathbb{C}$) that are:

1. of class C^∞ on Ω (all partial derivatives exist and are continuous);
2. with compact support in Ω .

Notation 9 We denote $C_c^\infty(\Omega) = \mathcal{D}(\Omega)$, and we write

$$\mathcal{D}(\Omega) = \{\varphi \in C^\infty(\Omega) : \text{supp}(\varphi) \text{ is compact in } \Omega\}.$$

Remark 10 The set $C_c^\infty(\Omega) = \mathcal{D}(\Omega)$ is also called the base space, and its elements are called test functions.

Remark 11 Saying that φ has compact support in Ω means: there exists a compact set $K \subset \Omega$ such that

$$\text{supp}(\varphi) \subset K.$$

Remark 12 The function φ vanishes on the boundary of Ω (on $\partial\Omega$).

Examples of functions that belong to $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$

Example 13 The function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$\varphi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1 \end{cases}$$

belongs to $\mathcal{D}(\mathbb{R})$.

To prove that $\varphi \in \mathcal{D}(\mathbb{R})$, we need to check two points.

1. $\varphi \in C^\infty(\mathbb{R})$. We distinguish three cases:

(a) if $|x| > 1$, then $\varphi(x) = 0$, hence $\varphi \in C^\infty(\mathbb{R})$.

(b) If $|x| < 1$, then $\frac{1}{1-x^2} \in C^\infty(\mathbb{R})$ and

$$\varphi(x) = \exp\left(-\frac{1}{1-x^2}\right) \in C^\infty(\mathbb{R}).$$

(c) if $|x| = 1$, we use the same reasoning as in the previous example; we show that

$$\varphi^{(k)}(x) = 0, \quad \forall k \in \mathbb{N}.$$

Then

$$\varphi \in C^\infty(\mathbb{R}).$$

2. $\text{supp}(\varphi)$ is compact in $\mathbb{R}$.

By definition,

$$\begin{aligned} \text{supp}(\varphi) &= \overline{\{x \in \mathbb{R} : \varphi(x) = 0\}} = \overline{\{x \in \mathbb{R} : |x| < 1\}} \\ &= \{x \in \mathbb{R} : |x| \leq 1\} = [-1, 1]. \end{aligned}$$

Observe that $\text{supp}(\varphi) = [-1, 1]$ is closed and it is also bounded and contained in $\mathbb{R}$. Then $\text{supp}(\varphi)$ is compact in $\mathbb{R}$.

Consequently

$$\varphi \in \mathcal{D}(\mathbb{R}).$$

Example 14 The zero constant function always belongs to $\mathcal{D}(\Omega)$, since its support is the empty set ϕ , which is considered to be a compact set contained in any set.

Examples of functions that do not belong to $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$

Example 15 The function φ defined on $\mathbb{R}$ by

$$x \mapsto \varphi(x) = 1$$

does not belong to $\mathcal{D}(\mathbb{R})$, since we have

$$\text{supp}(\varphi) = \overline{\{x \in \mathbb{R} : \varphi(x) \neq 0\}} = \overline{\mathbb{R}} = \mathbb{R}.$$

Since $\text{supp}(\varphi) = \mathbb{R}$ is not compact (it is not bounded), we have

$$\varphi \notin \mathcal{D}(\mathbb{R}).$$

Example 16 Consider the function $\mathbf{1}_{\mathbb{Q}}$ defined on $\mathbb{R}$ by

$$\mathbf{1}_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

We have

$$\text{supp}(\mathbf{1}_{\mathbb{Q}}) = \overline{\{x \in \mathbb{R} : \varphi(x) \neq 0\}} = \overline{\mathbb{Q}}.$$

Since $\overline{\mathbb{Q}} = \mathbb{R}$, we get

$$\text{supp}(\mathbf{1}_{\mathbb{Q}}) = \mathbb{R}$$

which is not compact. Therefore,

$$\mathbf{1}_{\mathbb{Q}} \notin \mathcal{D}(\mathbb{R}).$$

1.2 Convergence in $\mathcal{D}(\Omega)$

1.2.1 Some notations

Multi-index α

Definition 17 (Multi-index α) A multi-index α is a tuple of integers,

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$$

where each $\alpha_i \in \mathbb{N} = \{0, 1, 2, \dots\}$ (non-negative integers).

Definition 18 The order of α , denoted by $|\alpha|$, is the integer given by

$$|\alpha| = \alpha_1 + \dots + \alpha_n = \sum_{i=1}^n \alpha_i.$$

Example 19 Let $x = (x_1, x_2) \in \mathbb{R}^2$. For $\alpha = (2, 3) \in \mathbb{N}^2$, we write

$$x^\alpha = x_1^2 x_2^3.$$

Example 20 Let $x = (u, v, w) \in \mathbb{R}^3$. For $\alpha = (1, 0, 3) \in \mathbb{N}^3$, we write

$$x^\alpha = u^1 v^0 w^3 = uw^3.$$

Remark 21 If α and β are two multi-indices, then for every $i = 1, \dots, n$, we have the following equivalence:

$$\alpha \leq \beta \iff \alpha_i \leq \beta_i, \quad \text{for all } i = 1, \dots, n.$$

Remark 22 For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and the vectors $x = (x_1, \dots, x_n)^n$, $x = (y_1, \dots, y_n)^n$:

1. Factorial

$$\alpha! = \alpha_1! \dots \alpha_n!.$$

2. Power of a vector

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

3. Binomial formula

$$(x + y)^\alpha = \sum_{\beta \leq \alpha} C_\beta^\alpha x^{\alpha-\beta} y^\beta$$

$$\text{where } C_\beta^\alpha = \frac{\alpha!}{\beta! (\alpha - \beta)!}.$$

Derivative of order α

Definition 23 (Derivative of order α) Soit $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ et $\alpha \in \mathbb{N}^n$. We define

$$D^\alpha f = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}.$$

Remark 24 If $\alpha = (0, \dots, 0, \dots, 0)$. Then,

$$D^\alpha f = f.$$

Proposition 25 (Leibniz's formula) Let f and $g : \mathbb{R}^n \longrightarrow \mathbb{C}$. we have

$$D^\alpha (fg) = \sum_{\beta \leq \alpha} C_\beta^\alpha D^{\alpha-\beta} f D^\beta g.$$

1.2.2 Convergence in $\mathcal{D}(\Omega)$

Definition 26 Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence of functions in $\mathcal{D}(\Omega)$. We say that the sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges to φ in $\mathcal{D}(\Omega)$ if:

1. Supports uniformly contained in a compact set: there exists a compact set $K \subset \Omega$ such that

$$\text{supp}(\varphi_n) \subset K, \text{ for all } n \in \mathbb{N}$$

and $\text{supp}(\varphi) \subset K$.

2. Uniform convergence of all derivatives on K : for every multi-index α ,

$$\sup_{x \in K} |D^\alpha \varphi_n(x) - D^\alpha \varphi(x)| \xrightarrow{n \rightarrow +\infty} 0.$$

We write:

$$\varphi_n \longrightarrow \varphi \text{ in } \mathcal{D}(\Omega).$$

Exercise 27 Let $\Omega = \mathbb{R}$. Let $\varphi \in \mathcal{D}(\mathbb{R})$. Study the convergence in $\mathcal{D}(\mathbb{R})$ of the following sequence:

$$\psi_n(x) = \frac{1}{n} \varphi(x).$$

Solution 28 Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then there exists $a > 0$ such that

$$\text{supp}(\varphi) \subset [-a, a] = K \subset \mathbb{R}.$$

To study the convergence of the sequence $(\psi_n)_n$ in $\mathcal{D}(\mathbb{R})$:

1. Firstly, we begin by studying the pointwise convergence of $(\psi_n)_n$ in $\mathcal{D}(\mathbb{R})$.

Let x be fixed in $\mathbb{R}$, we have

$$\lim_{n \rightarrow +\infty} \psi_n(x) = \lim_{n \rightarrow +\infty} \frac{1}{n} \varphi(x) = 0.$$

Hence, the sequence $\psi_n(x)$ converges pointwise to $\psi(x) = 0$.

Observe that

$$\text{supp}(\psi_n) = \text{supp}(\varphi).$$

We have :

$$(a) \text{ supp}(\psi_n) \subset K = [-a, a];$$

$$(b) \text{ supp}(\psi) = \phi \subset K = [-a, a];$$

2. Secondly, from the pointwise convergence result, for every $\alpha \in \mathbb{N}^n$, we have

$$\sup_{x \in K} |D^\alpha \psi_n(x) - D^\alpha \psi(x)| = \sup_{x \in K} \left| \frac{1}{n} D^\alpha \varphi(x) - 0 \right|$$

Since $\frac{1}{n} > 0$,

$$\sup_{x \in K} |D^\alpha \psi_n(x) - D^\alpha \psi(x)| = \frac{1}{n} \sup_{x \in K} |D^\alpha \varphi(x)|$$

Since $\varphi \in C^\infty(\Omega)$ and K is compact, then $D^\alpha \varphi$ is continuous on K , hence bounded.

Thus

$$\sup_{x \in K} |D^\alpha \varphi(x)| < +\infty$$

Consequently,

$$\sup_{x \in K} |D^\alpha \psi_n(x) - D^\alpha \psi(x)| \xrightarrow{n \rightarrow +\infty} 0.$$

which is exactly the definition of uniform convergence of the derivatives on K .

General conclusion: From 1 and 2, we indeed have

$$\psi_n \longrightarrow 0 \text{ in } \mathcal{D}(\Omega).$$

1.3 Regularizing sequences

1.3.1 Definition of a regularizing sequence

Definition 29 Let $(\varphi_k)_{k \in \mathbb{N}}$ be a sequence of functions in $\mathcal{D}(\mathbb{R}^n)$. The sequence (φ_k) is called a regularizing sequence (or mollifying sequence) if it satisfies the following three conditions:

1. $\varphi_k \geq 0$,
2. $\int_{\mathbb{R}^n} \varphi_k(x) dx = 1$,
3. There exists a sequence of positive real numbers $r_k \xrightarrow{k \rightarrow +\infty} 0$ such that

$$\text{supp}(\varphi_k) \subset \overline{B(0, r_k)}.$$

Example 30 Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be defined by:

$$\varphi(x) = \begin{cases} C \exp \frac{1}{|x|^2 - 1}, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}$$

where $C > 0$ is chosen so that

$$\int_{\mathbb{R}^n} \varphi(x) dx = 1.$$

Observe that:

- $\varphi \in C^\infty$,
- $\varphi(x) \geq 0$,
- $\text{supp}(\varphi) = \overline{B(0, 1)}$

- has total integral 1.
For each $k \in \mathbb{N}$, define

$$\varphi_k(x) = k^n \varphi(kx).$$

Observe that:

1. Since $\varphi(x) \geq 0$ and $k^n > 0$, we have

$$\varphi_k(x) \geq 0, \quad \text{for all } x \in \mathbb{R}^n.$$

2. By the change of variables $y = kx$ (so $dx = (\frac{1}{k})^n dy$), we get

$$\int_{\mathbb{R}^n} \varphi_k(x) dx = \int_{\mathbb{R}^n} k^n \varphi(kx) dx = \int_{\mathbb{R}^n} \varphi(y) dy = 1.$$

3. Observe that $\varphi(kx) \neq 0$ only when

$$|kx| < 1$$

that is

$$|x| < \frac{1}{k}.$$

Then

$$\text{supp}(\varphi_k) \subset \overline{B\left(0, \frac{1}{k}\right)}.$$

So take $r_k = \frac{1}{k}$. Then

$$\lim_{k \rightarrow +\infty} r_k = 0.$$

Therefore (φ_k) is a regularizing sequence.

1.4 Partition of unity

1.4.1 Bump functions

Lemma 31 (Urysohn's lemma) Let K be a compact set in $\mathbb{R}^n$ and let F be a closed set in $\mathbb{R}^n$ such that

$$K \cap F = \emptyset.$$

Then there exists $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that

1. $0 \leq \varphi(x) \leq 1$, for all $x \in \mathbb{R}^n$,
2. $\varphi(x) = 1$, for all $x \in K$,
3. $\varphi(x) = 0$, for all $x \in F$.

Corollary 32 *Let Ω be an open subset of $\mathbb{R}^n$ and let K be a compact set of Ω . Then there exists $\varphi \in \mathcal{D}(\Omega)$ such that*

$$0 \leq \varphi(x) \leq 1,$$

and

$$\varphi(x) = 1 \text{ on } K.$$

Proof. Let Ω be an open subset of $\mathbb{R}^n$ and let K be a compact set of Ω . We want to construct $\varphi \in \mathcal{D}(\Omega)$ such that $0 \leq \varphi(x) \leq 1$, and $\varphi(x) = 1$ on K .

Let us

$$F = \mathbb{R}^n \setminus \Omega.$$

Then F is closed in $\mathbb{R}^n$. Since $K \subset \Omega$, we immediately have

$$K \cap F = \emptyset.$$

We are therefore under the hypotheses of Urysohn's lemma ($\mathbb{R}^n$ version).

By the lemma, there exists $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that

1. $0 \leq \varphi(x) \leq 1$, for all $x \in \mathbb{R}^n$,
2. $\varphi(x) = 1$, for all $x \in K$,
3. $\varphi(x) = 0$, for all $x \in F$.

Let us verify that $\varphi \in \mathcal{D}(\Omega)$

By (3), φ is identically zero on $C_{\mathbb{R}^n}\Omega$. In particular, the support of φ is contained in Ω :

$$\text{supp}(\varphi) \subset \Omega.$$

Indeed, if $x \notin \Omega$, then there exists a neighborhood (for example $\mathbb{R}^n \setminus \Omega$ itself) on which

$$\varphi = 0,$$

hence

$$x \notin \text{supp}(\varphi).$$

Since $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\text{supp}(\varphi) \subset \Omega$, the restriction of φ to Ω belongs to $C_c^\infty(\Omega)$, that is,

$$\varphi \in \mathcal{D}(\Omega).$$

■

Definition of a bump function

Definition 33 *Let Ω be an open subset of $\mathbb{R}^n$ and let K be a compact set of Ω . A bump function is defined as any function $\varphi \in C^\infty(\Omega)$ satisfying:*

1. $\text{supp}(\varphi) \subset \Omega$,
2. $0 \leq \varphi(x) \leq 1$, for all $x \in \Omega$,
3. $\varphi(x) = 1$, for all $x \in K$.

1.4.2 Partition of unity

Theorem 34 (finite partition of unity on a compact) *Let K be a compact set of $\mathbb{R}^n$ and let $\{\Omega_k\}_{k=1}^N$ family of open sets such that*

$$K \subset \Omega_1 \cup \dots \cup \Omega_N = \Omega.$$

Then there exist functions $\varphi_1, \dots, \varphi_N \in \mathcal{D}(\Omega_k)$ such that

1. $0 \leq \varphi_k(x) \leq 1$,
2. $\text{supp}(\varphi_k) \subset \Omega_k$,
3. $\sum_{k=1}^N \varphi_k(x) = 1$ on a neighborhood of K .

1.5 Exercises

Exercise 35 *Determine the support of the following functions:*

In $\mathbb{R}$

1. $H(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$ (the Heaviside function).
2. $a(x) = \begin{cases} 0, & \text{for all } x \leq 0, \\ \exp\left(\frac{-1}{x}\right), & \text{for all } x > 0. \end{cases}$
3. $b(x) = a(x) - a(1-x).$
4. $\theta_\alpha(x) = \frac{a(x)}{a(x) + a(\alpha - x)} = \begin{cases} 1, & \text{if } x \in [\alpha, +\infty[, \\ 0, & \text{if } x \in]-\infty, 0]. \end{cases}$
5. $f(x) = \mathbf{1}_{\mathbb{R}_+^*}(x)$ et $g(x) = \mathbf{1}_{\mathbb{R}_-^*}(x).$

In $\mathbb{R}^n$

6. $\rho(x) = \begin{cases} 0, & \text{si } |x| \geq r \left(|x| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}} \right), \\ \exp\left(\frac{-\rho^2}{r^2 - |x|^2}\right), & \text{for all } |x| < r. \end{cases}$

Exercise 36 *Show that if f and g are two functions from Ω to $\mathbb{R}$ then:*

$$\text{supp}(fg) \subset \text{supp}(f) \cap \text{supp}(g).$$

Exercise 37 Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a function of class $C^\infty(\mathbb{R})$. Show that if $\varphi \in \mathcal{D}(\mathbb{R})$, then $f\varphi \in \mathcal{D}(\mathbb{R})$.

Exercise 38 Let $\varphi \in \mathcal{D}(\mathbb{R}^*)$ and let the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ define by

$$f(x) = |x| \varphi(x).$$

Show that $f \in \mathcal{D}(\mathbb{R})$.

Exercise 39 Show that the function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\psi(t) = \begin{cases} \exp\left(\frac{-1}{t}\right), & \text{si } t > 0, \\ 0, & \text{si } t \leq 0. \end{cases}$$

is of class $C^\infty(\mathbb{R})$.

Deduce that the function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} \exp\left(\frac{-1}{1-|x|^2}\right), & \text{si } |x| < 1, \\ 0, & \text{si } |x| \geq 1. \end{cases}$$

belongs to $\mathcal{D}(\mathbb{R}^n)$.

Exercise 40 Let the function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} \exp\left(\frac{-1}{1-|x|^2}\right), & \text{si } |x| < 1, \\ 0, & \text{si } |x| \geq 1. \end{cases}$$

1. Show that $\varphi \in \mathcal{D}(\mathbb{R})$.

2. Let us

$$\psi(x) = \int_{-\infty}^x \varphi(y) dy.$$

Does $\psi \in C^\infty(\mathbb{R})$? Determine its support.

3. Let us

$$f(x) = \varphi(x) - \varphi(x-3)$$

and

$$g(x) = \int_{-\infty}^x f(y) dy.$$

Does $g \in C^\infty(\mathbb{R})$? Determine its support.

Exercise 41 Let $\varphi \in \mathcal{D}(\mathbb{R})$. Let us

$$\varphi_n(x) = n \left[\varphi \left(x + \frac{1}{n} \right) - \varphi(x) \right], \text{ pour tout } n \in \mathbb{N}^*.$$

1. Show that $\varphi_n \in \mathcal{D}(\mathbb{R})$.
2. Show that as $n \longrightarrow +\infty$, φ_n converges pointwise to φ' .