

Chapter 1

Topological Spaces

1.1 Topological spaces, Discrete topology, and indiscrete topology

Let X be a non-empty set.

1.1.1 Topology and Topological space

Topology

Definition 1 A topology on X is a collection τ of subsets of X (that is, $\tau \subset \mathcal{P}(X)$) satisfying the following Hausdorff axioms:

Axiom (O_1) : $X, \phi \in \tau$.

Axiom (O_2) : The union of the elements of any (**finite or infinite**) subcollection of τ is in τ .

On the other word: for any family $(U_i)_{i \in I}$ of sets in τ ,

$$\bigcup_{i \in I} U_i \in \tau.$$

Axiom (O_3) : The intersection of the elements of any finite subcollection of τ is in τ .

On the other word: for any **finite** family $(U_i)_{1 \leq i \leq n}$ of sets in τ ,

$$\bigcap_{i=1}^n U_i \in \tau.$$

Topological space

Definition 2 The pair (X, τ) is called a topological space.

Examples

Example 3 Let $X = \{a, b, c, d, e, f\}$ and

$$\tau_1 = \{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}.$$

Then τ_1 is a topology on X as it satisfies conditions (O_1) , (O_2) , and (O_3) of definitions 1.

The couple (X, τ_1) is a **topological space**.

Example 4 Let $X = \{a, b, c, d, e\}$ and

$$\tau_2 = \{X, \phi, \{a\}, \{c, d\}, \{a, c, e\}, \{b, c, d\}\}.$$

Then τ_2 is not a topology on X since the union

$$\underbrace{\{c, d\}}_{\in \tau_2} \cup \underbrace{\{a, c, e\}}_{\in \tau_2} = \{a, c, d, e\} \notin \tau_2,$$

that is, τ_2 does not satisfy condition (O_2) of definitions 1.

Example 5 Let $X = \{a, b, c, d, e, f\}$ and

$$\tau_3 = \{X, \phi, \{a\}, \{f\}, \{a, f\}, \{a, c, f\}, \{b, c, d, e, f\}\}.$$

Then τ_3 is not a topology on X since the intersection

$$\underbrace{\{a, c, f\}}_{\in \tau_3} \cap \underbrace{\{b, c, d, e, f\}}_{\in \tau_3} = \{c, f\} \notin \tau_3,$$

that is, τ_3 does not satisfy condition (O_3) of definitions 1.

Example 6 Let $\mathbb{N}$ be the set of all natural numbers and let

$$\tau_4 = \{\phi, \mathbb{N}\} \cup \{S \subset \mathbb{N} : S \text{ is finite}\}.$$

Then τ_4 is not a topology on $\mathbb{N}$, since the infinite union

$$\underbrace{\{2\}}_{\substack{\text{finite} \\ \in \tau_4}} \cup \underbrace{\{3\}}_{\substack{\text{finite} \\ \in \tau_4}} \cup \dots \cup \underbrace{\{n\}}_{\substack{\text{finite} \\ \in \tau_4}} \cup \dots = \underbrace{\{2, 3, \dots, n, \dots\}}_{\text{infinite set}} \notin \tau_4$$

that is, τ_4 does not satisfy condition (O_2) of definitions 1.

1.1.2 Discrete topological space

Discrete topology

Definition 7 The discrete topology τ_d on X is the topology consisting of all subsets of X .

That is $\tau_d = \mathcal{P}(X)$.

Discrete space

Definition 8 The pair (X, τ_d) is called a discrete space.

Exercise 9 Let $X = \{a, b, c\}$ and let τ be a topology on X with $\{a\} \in \tau$, $\{b\} \in \tau$, and $\{c\} \in \tau$.

Prove that τ is the discrete topology.

Solution 10 We are given a topology τ on X , and we know that $\{a\} \in \tau$, $\{b\} \in \tau$, and $\{c\} \in \tau$.

We need to show that τ is the discrete topology.

For this end, we need to show that every subset of X belongs to τ .

Indeed, since τ is a topology, it satisfies (O_1) , (O_2) , and (O_3) .

The set X has 3 elements, then the cardinality of X is given by:

$$|X| = n(X) = \text{card}(X) = 3,$$

so the cardinality of $\mathcal{P}(X)$ is

$$\text{card}(\mathcal{P}(X)) = 2^3 = 8.$$

Then

$$\mathcal{P}(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$

Set $S_1 = \phi$, $S_2 = \{a\}$, $S_3 = \{b\}$, $S_4 = \{c\}$, $S_5 = \{a, b\}$, $S_6 = \{a, c\}$, $S_7 = \{b, c\}$, $S_8 = X$

Now, we need to show that each of these subsets is in τ .

For S_1, S_8

Since τ is a topology on X , then by the topology axioms (O_1) , $\phi, X \in \tau$. That is, $S_1, S_8 \in \tau$.

For S_2, S_3, S_4

We are given that $\{a\} \in \tau$, $\{b\} \in \tau$, and $\{c\} \in \tau$. That is, $S_2 \in \tau$, $S_3 \in \tau$, and $S_4 \in \tau$.

For S_5

We observe that

$$S_5 = \{a, b\} = \{a\} \cup \{b\} = S_2 \cup S_3.$$

Since $S_2, S_3 \in \tau$, their union $S_2 \cup S_3 \in \tau$. Hence $S_5 = \{a, b\} \in \tau$.

For S_6

We have

$$S_6 = \{a, c\} = \{a\} \cup \{c\} = S_2 \cup S_4.$$

Since $S_2, S_4 \in \tau$, their union $S_2 \cup S_4 \in \tau$. Hence $S_6 = \{a, c\} \in \tau$.

For S_7

We have

$$S_7 = \{b, c\} = \{b\} \cup \{c\} = S_3 \cup S_4.$$

Since $S_3, S_4 \in \tau$, their union $S_3 \cup S_4 \in \tau$. Hence $S_7 = \{b, c\} \in \tau$.

Consequently, every subset of X belongs to τ , so τ is the discrete topology.

Proposition 11 *Let (X, τ) be a topological space. If for every $x \in X$, the singleton set $\{x\} \in \tau$, then τ is the discrete topology.*

Proof. Note that every set is the union of all its singleton subsets.

On other word: Let $S \subset X$. Then

$$S = \bigcup_{x \in X} \{x\}.$$

Since $\{x\} \in \tau$, and since τ is a topology on X , it follows $\bigcup_{x \in X} \{x\} \in \tau$.

Therefore,

$$S \in \tau.$$

Because S was an arbitrary subset of X (i.e., $S \in \mathcal{P}(X)$), every subset of X belongs to τ .

Hence τ is the discrete topology. ■

1.1.3 Indiscrete topological space

Trivial topology

Definition 12 *Let $\tau_{tri} = \{X, \phi\}$. Then τ_{tri} is called the trivial topology (also known as the indiscrete topology).*

Indiscrete space

Definition 13 and the pair (X, τ_{tri}) is called an indiscrete space (or a trivial space).

1.2 Open Sets, Closed Sets, and Clopen Sets**1.2.1 Open Sets**

Definition 14 Let (X, τ) be a topological space. Then the element of τ are called open sets.

Example 15 Let $X = \{a, b\}$. There are several possible topologies on X . Indeed, since the cardinality of X is

$$n(X) = 2,$$

the cardinality of its power set $\mathcal{P}(X)$ is

$$n(\mathcal{P}(X)) = 2^2 = 4.$$

Thus

$$\mathcal{P}(X) = \{\phi, \{a\}, \{b\}, X\}.$$

Consequently, the topologies on X are as follows:

1. $\tau_1 = \tau_{tri} = \{\phi, X\}$, so ϕ, X are the only open sets.
2. $\tau_2 = \mathcal{P}(X) = \tau_d$, so every element of $\mathcal{P}(X)$ is an open set.
3. $\tau_3 = \{\phi, \{a\}, X\}$, so $\phi, \{a\}, X$ are all open sets.
4. $\tau_4 = \{\phi, \{b\}, X\}$. so $\phi, \{b\}, X$ are all open sets.

Remark 16 The axioms (O_1) , (O_2) , and (O_3) can be reformulated as follows:

Axiom (O_1) : X and ϕ are open sets.

Axiom (O_2) : The union of any family (**finite or infinite**) of open sets is open.

Axiom (O_3) : The intersection of any **finite** family of open sets is open.

Example 17 Let $\mathbb{N}$ be the set of all positive integers and let

$$\tau = \{\phi\} \cup \{D \subset \mathbb{N} : \mathbb{N} \setminus D, \text{ is finite}\}.$$

It is easily verified that τ satisfies the Hausdorff axioms, (O_1) , (O_2) , and (O_3) and is therefore a topology on $\mathbb{N}$.

For $n \in \mathbb{N}$, define the set D_n as follows:

$$D_n = \{1\} \cup \{n+1\} \cup \{n+2\} \cup \{n+3\} \cup \dots = \{1\} \cup \bigcup_{m=n+1}^{+\infty} \{m\}.$$

Observe that for example:

For $n = 1$:

$$D_1 = \{1\} \cup \{2\} \cup \{3\} \cup \dots = \{1, 2, 3, \dots\} = \mathbb{N}, \text{ so}$$

$$\mathbb{N} \setminus D_1 = \phi \text{ finite}$$

thus $D_1 \in \tau$ (*that is D_1 is open*) ..

For $n = 2$:

$$D_2 = \{1\} \cup \{3\} \cup \{4\} \cup \dots = \{1, 3, 4, \dots\}, \text{ so}$$

$$\mathbb{N} \setminus D_2 = \{2\} \text{ finite}$$

thus $D_2 \in \tau$ (*that is D_2 is open*) ..

$$\text{For } n = 3 : D_3 = \{1\} \cup \{4\} \cup \{5\} \cup \dots = \{1, 4, 5, \dots\}, \text{ so}$$

$$\mathbb{N} \setminus D_3 = \{2, 3\} \text{ finite}$$

thus $D_3 \in \tau$ (*that is D_3 is open*) ..

Clearly each D_n is an open set in the topology τ , since its complement $\mathbb{N} \setminus D_n$ is a finite set.

However,

$$\begin{aligned} \bigcap_{n=1}^{+\infty} D_n &= D_1 \cap D_2 \cap D_3 \cap \dots \cap D_n \cap \dots \\ &= \{1\}. \end{aligned}$$

Observe that

$$\mathbb{N} \setminus \{1\} = \{2, 3, 4, \dots\}.$$

Since $\mathbb{N} \setminus \{1\}$ is neither $\mathbb{N}$ nor a finite set, $\{1\}$ is not open.

Thus, this shows that the intersection of the open sets D_n is not open.

1.2.2 Closed Sets

Definition 18 Let (X, τ) be a topological space. A subset S of X is said to be a closed set in (X, τ) if its complement in X , namely $X \setminus S$, is open in (X, τ) .

Example 19 Let $X = \{a, b, c\}$ and

$$\tau = \{ \phi, \{a\}, \{a, b\}, \{a, c\}, X \}$$

be a topology on X .

1. The open sets are:

- (a) ϕ .
- (b) $\{a\}$.
- (c) $\{a, b\}$.
- (d) $\{a, c\}$.
- (e) X .

2. The closed sets are complements:

- (a) $X - \phi = X$.
- (b) $X - \{a\} = \{b, c\}$.
- (c) $X - \{a, b\} = \{c\}$.
- (d) $X - \{a, c\} = \{b\}$.
- (e) $X - X = \phi$.

Example 20 If (X, τ_d) is a discrete space, then every subset of X is a closed set.

Example 21 If (X, τ_{tri}) is a trivial space, the only closed sets are X and ϕ .

Proposition 22 If (X, τ) is any topological space, then

- 1. ϕ and X are closed sets,

2. the **intersection** of any family (**finite or infinite**) of closed sets is a closed set,
3. the **union** of any **finite** family of closed sets is a closed set.

Remark 23 A set can be both open and closed, or neither open nor closed.

Example 24 In a discrete space every set is both open and closed.

Example 25 In an indiscrete space (X, τ_{tri}) , all subsets of X except X and ϕ are neither open nor closed.

1.2.3 Clopen Sets

Definition 26 A subset S of a topological space (X, τ) is said to be clopen if it is both open and closed in (X, τ) .

Example 27 In any topological space (X, τ) , both X and ϕ are clopen.

Example 28 In a discrete space, every subsets of X is clopen.

Example 29 In an indiscrete space, the only clopen subsets are X and ϕ .

1.3 The Finite-Closed Topology

It is usual to define a topology on a set by specifying which sets are open. However, it is sometimes more natural to describe the topology by specifying which sets are closed.

1.3.1 The Cofinite Topology

Definition 30 A topology τ on X is called the cofinite topology (or the finite-closed topology) if the closed subsets of X are X and all finite subsets of X .

That is, the open sets are ϕ and all subsets of X whose complements are finite.

Remark 31 Note that this definition does not say that every topology which has X and the finite subsets of X closed is the finite-closed topology. These must be the only closed sets.

Example 32 *In the discrete topology, the set X and all finite subsets of X are closed, but so too are all other subsets of X .*

Remark 33 *In the cofinite topology all finite sets are closed.*

Remark 34 *In the cofinite topology, the infinite subsets need not be open sets.*

Example 35 *Let $\mathbb{N}$ denote the set of all positive integers. In the finite-closed topology, every finite set is closed. Therefore, sets such as*

$$\{1\}, \{5, 6, 7\}, \{2, 4, 6, 8\}$$

are closed.

Consequently, their complements

$$\{2, 3, 4, 5, \dots\}, \{1, 2, 3, 4, 8, 9, 10, \dots\}, \{1, 3, 5, 7, 9, 10, 11, \dots\}$$

are open.

On the other hand, the set of even positive integers is not a closed, because it is infinite. Hence its complement, the set of odd positive integers, is not open. This shows that while every finite set is closed, it is not true that every infinite set is open.

Exercise 36 *Let τ be the cofinite topology on X . Prove that if X has at least (au moins) three (03) distinct clopen subsets, then X is finite.*

Solution 37 *Recall that:*

1. *In the finite-closed topology, the closed sets are exactly X and all finite subsets of X .*
2. *A clopen set is one that is both closed and open.*
3. *Every topological space has at least two clopen sets: X and ϕ .*

We are given that (X, τ) is the cofinite topology and that it has at least three distinct clopen subsets.

Since there are at least three clopen subsets, there must exist a clopen set S with $S \neq X$ and $S \neq \phi$.

Because S is open, its complement $X \setminus S$ is closed. Thus S and $X \setminus S$ are closed in the finite-closed topology.

Thus both S and $X \setminus S$ are closed in the finite-closed topology.

Since neither S nor $X \setminus S$ equals X , both must be finite.

But

$$X = S \cup (X \setminus S)$$

so X is the union of two finite sets. Therefore X is finite.

Remark 38 *We now know three distinct topologies we can put on any infinite set and there are many more. The three we know are the discrete topology, the indiscrete topology, and the finite-closed topology. So we must be careful always to specify the topology on a set.*

Example 39 *The set $\{n : n \geq 10\}$ is open in the finite-closed topology on the set of natural numbers, but is not open in the indiscrete topology.*

The set of odd natural numbers is open in the discrete topology on the set of natural numbers, but is not open in the finite-closed topology.