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HEAT TRANSFER I

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Introduction

I. Introduction to Heat Transfer and Position vis-à-vis Thermodynamics

From thermodynamics, we have learned that energy can be transferred between a system and its environment. These interactions are called work and heat. However, thermodynamics concerns only the final states of the processes during which these interactions occur and provides no information regarding the nature of the interactions or their temporal rates.

In what follows, we will focus on extending thermodynamic analysis through the study of heat transfer modes and seek the necessary relationships for calculating heat transfer rates.

So what is heat transfer? The answer to such a question is the following: Heat transfer is the transfer of thermal energy due to spatial differences in temperature. In other words, wherever there is a temperature difference within a medium or between media, there will be a heat transfer.

The fields of thermodynamics and heat transfer are closely related; this concerns the rate at which heat is transferred, which can be considered as an extension of thermodynamics.

The first principle of thermodynamics is the principle of conservation of energy, which relates the different forms and interactions of energy between a system and its environment. Based on extensive experimental evidence, energy cannot be created or destroyed during a process; it can only change form. The first principle, or the first law of thermodynamics, expresses the fact that:

When a system undergoes a transformation, the algebraic sum of the various energy variations, heat flux, work performed, etc., is independent of this transformation. It depends solely on the initial state and the final state of the system.

The total energy E of a system is the sum of the kinetic energy E_C , the potential energy E_P , and the internal energy U

$$E = E_C + E_P + U \quad (I.1)$$

Kinetic and potential energies are common to both mechanics and

thermodynamics, but internal energy is a property specific to thermodynamics.

For a closed system, the total energy E exchanged with the environment is divided into two parts: the heat input Q to the system and the work performed W by the system. In this case, Equation I.1 is written as

$$\Delta E = Q + W \quad (I.2)$$

Generally, in thermodynamics, the variations in kinetic and potential energy are zero, and the previous equation becomes

$$\Delta U = Q + W \quad (I.3)$$

For an open system, the first principle of thermodynamics for a control volume can be written in the form

$$\Delta E = E_e - E_s + E_g \quad (I.4)$$

where E_e , E_s and E_g are respectively the incoming energy, the outgoing energy and the energy generated within the system. If we focus on the rates of energy exchange, we can rewrite relation I.4 in the form

$$\frac{dE}{dt} = \dot{E}_e - \dot{E}_s + \dot{E}_g \quad (I.5)$$

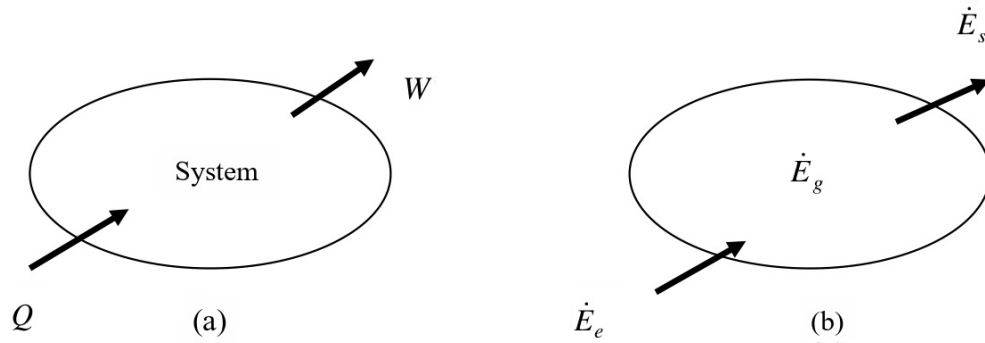


Figure I. 1. Conservation of energy: (a) for a closed system, (b) for an open system.

II. Phenomenological Laws of Heat Transfer

Heat transfer occurs through three different modes: conduction, convection, and thermal radiation. When a temperature gradient exists within a stationary medium, which may be a solid or a fluid, we use the term conduction to refer to heat transfer occurring through the medium. The term convection refers to the mode of heat transfer between a solid surface and a moving fluid or between two moving fluids when a temperature gradient exists. The third mode of heat transfer involves thermal radiation. Any solid surface or fluid emits thermal radiation in the form of electromagnetic waves (light) that travel without a material medium through the surrounding space.

II.1. Conduction

Conduction is the process by which heat is transmitted within a solid system from its hot region to its cold region or between two systems in contact (one hot and the other cold). This energy transfer occurs through the vibrations of the system's molecules around their equilibrium positions. According to kinetic theory, the energy or speed of the molecules is proportional to the temperature. Therefore, as they move toward regions of lower temperature, the molecules release part of their energy through elastic collisions or by electron diffusion (in the case of metals). Conduction tends to equalize the temperature within the system, but the process continues as long as a temperature difference is maintained.

Consider a solid system A formed by two subsystems A_1 and A_2 . Let S be the contact surface of the two subsystems which are at temperatures $T(M, t) + dT(M, t)$ and $T(M, t)$.

The amount of heat transferred dQ between subsystem A_1 and subsystem A_2 across the contact surface S with unit normal vector $\vec{n}$ is given by the formula

$$dQ = \vec{q} \cdot \vec{n} dS = q \cos \theta dS \quad (\text{I.6})$$

where θ is the angle between the direction of the heat flux $\vec{q}$ and the unit normal vector $\vec{n}$ of the surface.

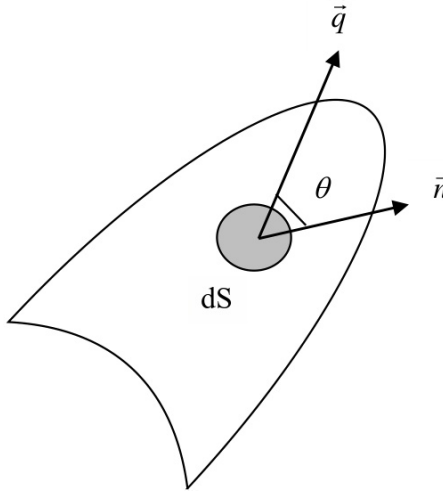


Figure I. 2. Definition of heat flux.

Points having the same temperature $T(M, t)$ form a surface called an isotherm. The heat flux traversing the surface S within the system is due to the temperature gradient between the isotherms. As this heat flux is caused

due to the temperature gradient, it is natural to consider a linear relationship between them. This postulates that the flux is directly proportional to the temperature gradient. This relationship can be written in the form

$$\vec{q} = -\lambda \vec{\nabla} T(M, t) \quad (I.7)$$

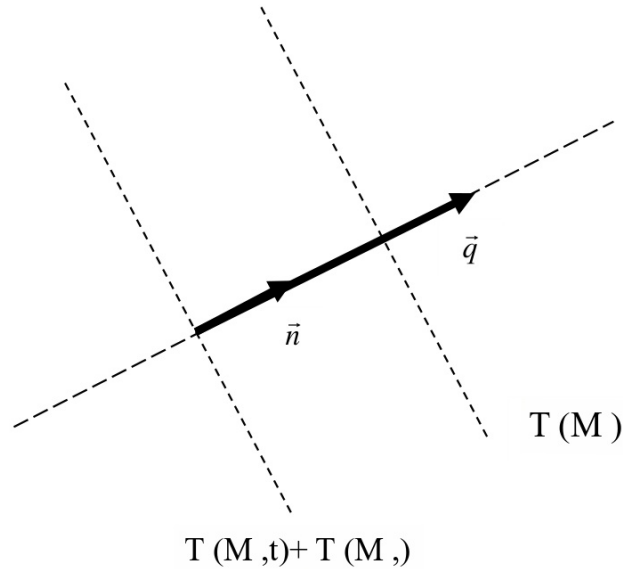


Figure I.3. Isotherms and Fourier's Law.

where λ is the thermal conductivity of the medium. Thermal conductivity is a physical property of media that characterizes their ability to transmit heat from one point to another.

This equation is Fourier's law. The minus sign indicates that heat always flows in the direction of decreasing temperatures, in accordance with the second law of thermodynamics. The proportionality constant is the thermal conductivity of the material; it is a function of temperature and pressure.

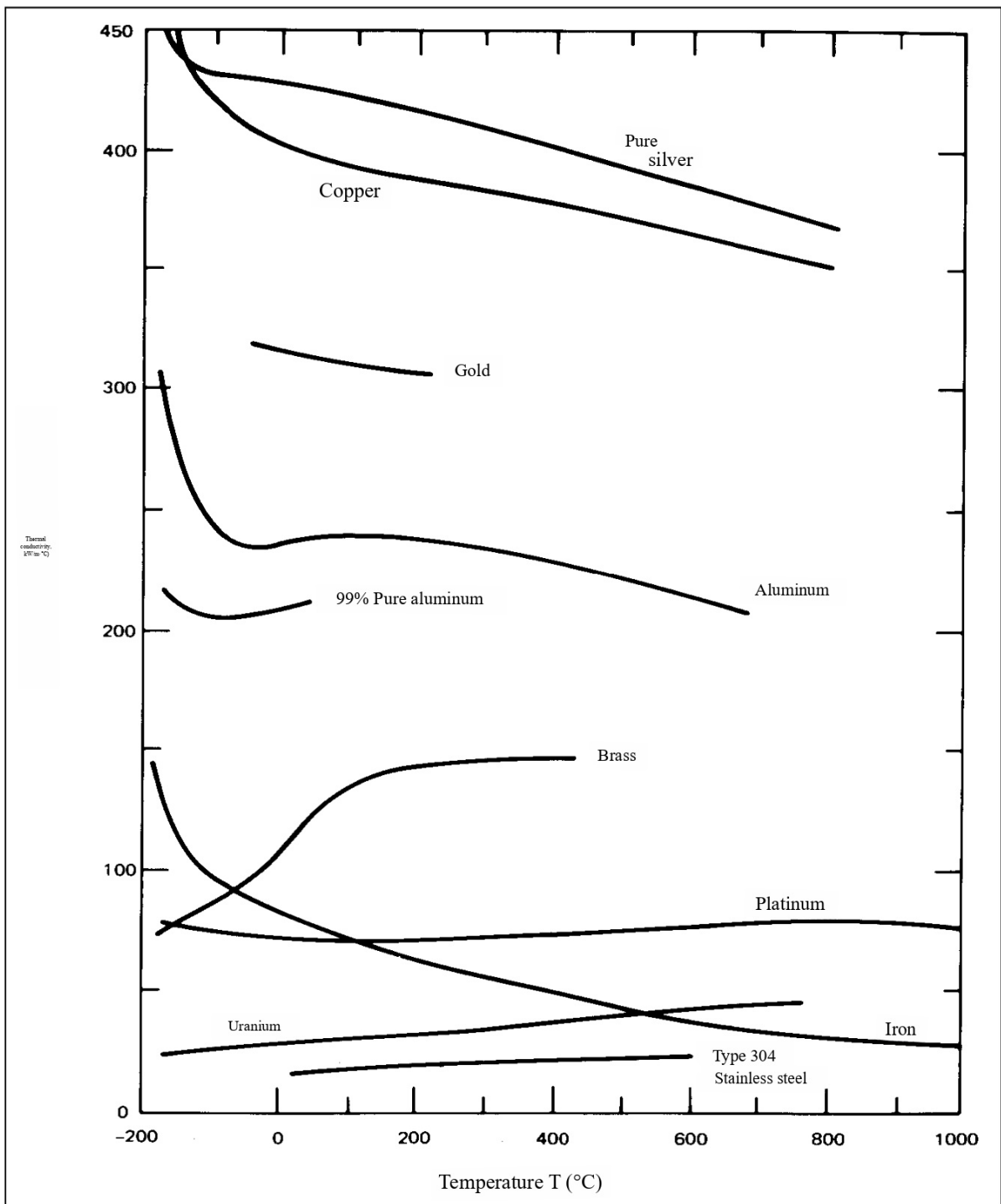


Figure I. 4. Variations in the thermal conductivity of solids as a function of temperature.

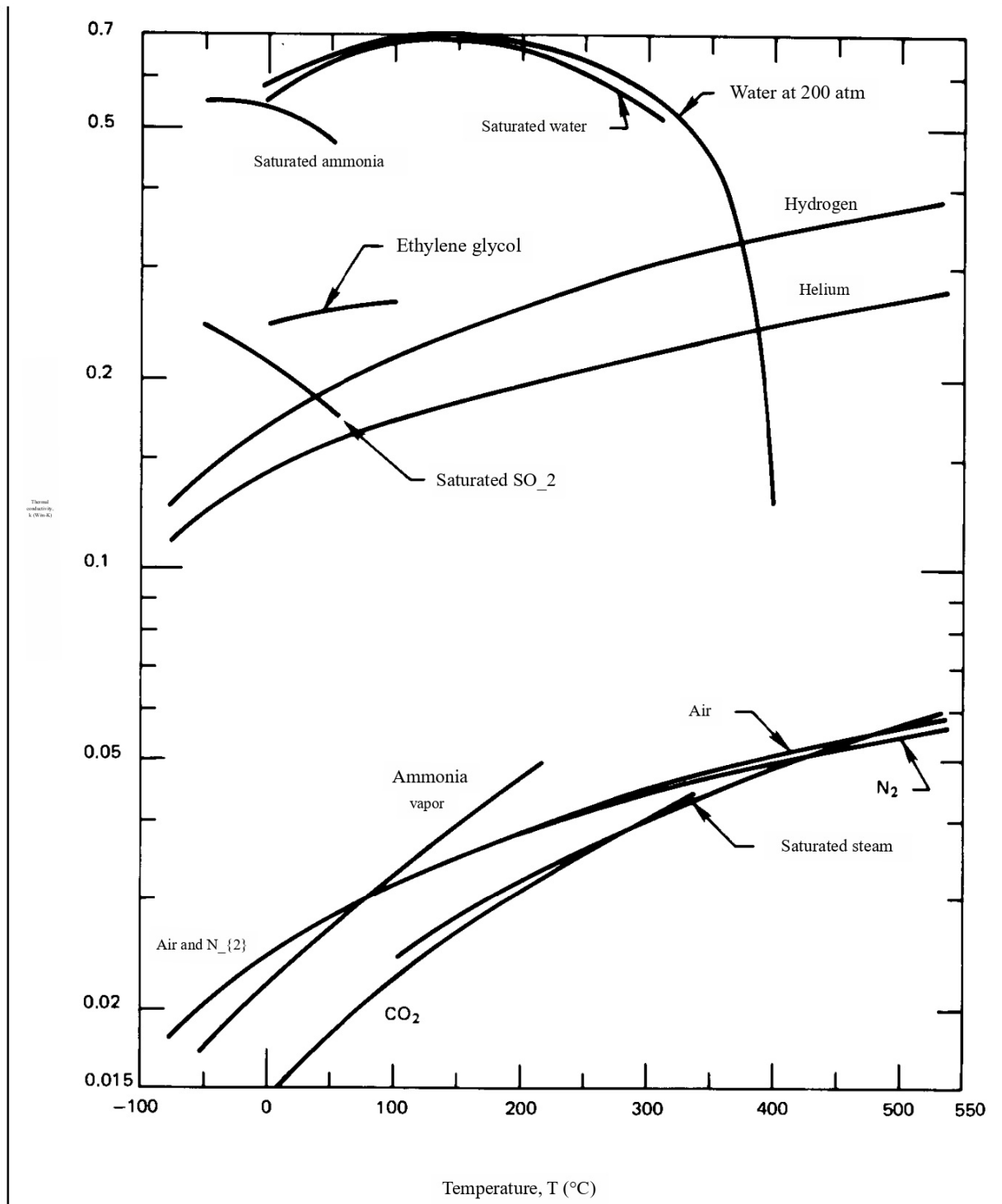


Figure I. 5. Variations in the thermal conductivity of certain gases as a function of temperature.

II. 2. Convection

During heat transfer between a fluid and a solid or between two fluids, the amount of heat exchanged is given by Newton's formula

$$Q = hS (T_0 - T_\infty) \quad (\text{I.8})$$

where h is the average convective heat transfer coefficient, T_0 and T_∞ are respectively the surface and fluid temperatures, and S is the exchange surface.

To calculate this coefficient, it is assumed that the quantity of heat on the surface occurs by conduction, such that

$$-\lambda \frac{\partial T}{\partial y} \Big|_{y=0} = h_x (T_0 - T_\infty) \quad (\text{I.9})$$

Thus, the local convective heat transfer coefficient h_x is

$$h_x = \frac{-\lambda \frac{\partial T}{\partial y} \Big|_{y=0}}{T_0 - T_\infty} \quad (\text{I.10})$$

The average convective heat transfer coefficient is calculated using the following formula

$$\bar{h} = \frac{1}{L} \int_0^L h_x dx \quad (\text{I.11})$$

The calculation of this convective heat transfer coefficient requires the solution of the fluid balance equations.

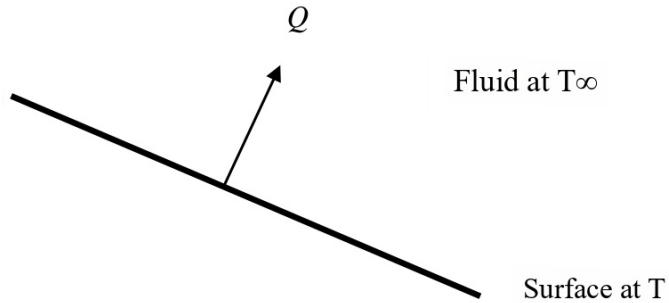


Figure I.6. Illustration of Newton's Law.

II. 3. Thermal Radiation

Every surface emits radiation known as thermal radiation, provided its temperature is different from absolute zero. This quantity of heat is calculated using the Stefan-Boltzmann formula, namely

$$Q = \varepsilon \sigma S T^4 \quad (\text{I.12})$$

where the coefficient ε is the emissivity of the surface, $\sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$ is the Stefan-Boltzmann constant, and S is the emitting surface. If the surface is black, this coefficient is equal to one.

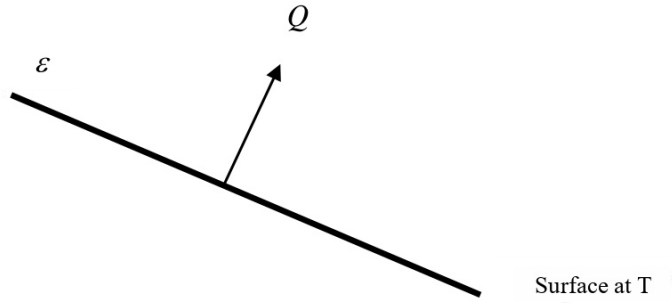


Figure I. 7. Illustration of the Stefan-Boltzmann law.

If the surface is surrounded by an infinite medium at temperature T_∞ , the net amount of heat exchanged is

$$Q = \varepsilon \sigma S (T_0^4 - T_\infty^4) \quad (\text{I.13})$$

If heat transfer occurs between two surfaces S_1 and S_2 , the net amount of heat exchanged between these two surfaces will be

$$Q = Q_{12} - Q_{21} = \sigma S_1 F_{12} (T_1^4 - T_2^4) \quad (\text{I.14})$$

The term F_{12} is called the shape factor; it depends on the emissivities of the two surfaces, the distance between the surfaces, and their respective orientations with respect to one another.

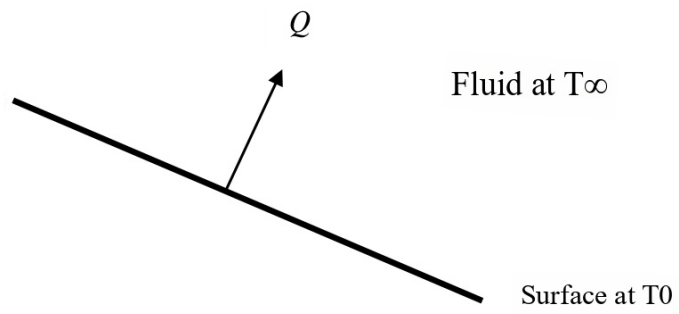


Figure I. 8. Illustration of Newton's Law.

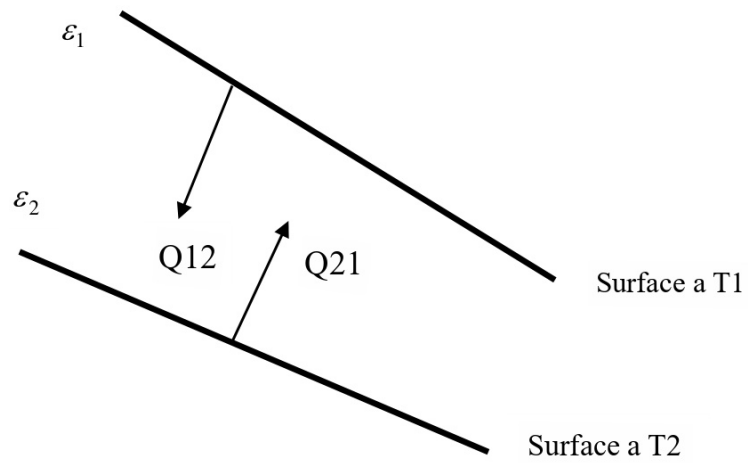


Figure I. 9. Heat exchange between two surfaces.

Conduction

II. Heat Equation

II.1. Derivation of the Heat Equation

Let us write an energy balance on a volume element v surrounded by a surface s .

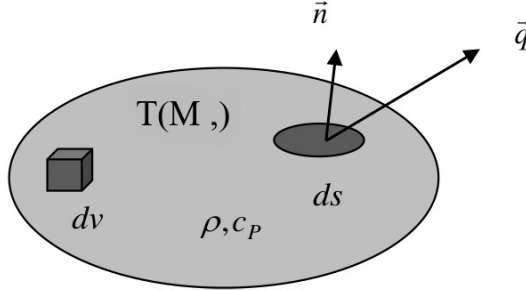


Figure II.1. Control volume in a heat transfer field.

The variation in internal energy U is equal to the sum of the heating power and the internal heat generated by an internal source of volumetric density q_s

$$\frac{dU}{dt} = - \int_s \vec{q} \cdot \vec{n} ds + \int_v q_s dv \quad (\text{II.1})$$

At the same time, the variation in internal energy, given that the latter depends solely on temperature, is expressed as

$$\frac{dU}{dt} = \int_v \rho C_p \frac{\partial T(M, t)}{\partial t} dv \quad (\text{II.2})$$

where ρ and C_p are respectively the volumetric mass and the specific heat capacity at constant pressure of the medium.

By inserting these terms into the energy balance, we obtain

$$\int_v \rho C_P \frac{\partial T(M,t)}{\partial t} dv = - \int_v \text{div} \vec{q} dv + \int_v q_s dv \quad (\text{II.3})$$

knowing that, from Fourier's law,

$$\vec{q} = -\lambda \vec{\nabla} T(M,t) \quad (\text{II.4})$$

we arrive at

$$\int_v \rho C_P \frac{\partial T(M,t)}{\partial t} dv = \int_v \text{div}(\lambda \vec{\nabla} T(M,t)) dv + \int_v q_s(M,t) dv \quad (\text{II.5})$$

by placing all terms under the integration sign, we obtain

$$\int_v \left[\rho C_P \frac{\partial T(M,t)}{\partial t} - \text{div}(\lambda \vec{\nabla} T(M,t)) - q_s(M,t) \right] dv = 0 \quad (\text{II.6})$$

this integral can be zero only if the function under the integral sign is zero, hence the relation

$$\rho C_P \frac{\partial T(M,t)}{\partial t} = \text{div}(\lambda \vec{\nabla} T(M,t)) + q_s(M,t) \quad (\text{II.7})$$

this is the heat equation in its general form within a stationary solid in the presence of internal heat sources. Generally, the thermal conductivity λ can be considered constant, and consequently, the heat equation takes the form

$$\rho C_P \frac{\partial T(M,t)}{\partial t} = \lambda \Delta T(M,z) + q_s(M,t) \quad (\text{II.8})$$

where Δ is the Laplacian operator.

Dividing both sides of this equation by the term ρC_P , we obtain

$$\frac{\partial T(M, t)}{\partial t} = a \Delta T(M, z) + \frac{q_s(M, t)}{\rho C_p} \quad (\text{II.9})$$

the coefficient $a = \frac{\lambda}{\rho C_p}$ is called the thermal diffusion coefficient.

II.2. Boundary Conditions

If we examine the heat equation carefully, we observe that it contains a first-order derivative with respect to time and second-order derivatives with respect to space; this necessarily implies three boundary conditions. There is one initial condition with respect to time and two boundary conditions with respect to space, which we will detail in the following.

II.2.1. Initial condition

If the temperature distribution at time $t = t_0$ is a certain function $f(M, t_0)$, the initial condition is

$$T(M, t) = f(M, t_0) \quad (\text{II.10})$$

II.2.2. Boundary Conditions

The boundary conditions imposed on the temperature $T(M, t)$ depend on the physical scenarios considered.

a) Surface at constant temperature (Dirichlet condition) If, for a spatial position M_0 , the temperature is constant, then the boundary condition in this case is

$$T(M, t) = g(M_0, t) \quad (\text{II.11})$$

This condition is also called a first-type condition. In practice, this boundary condition occurs in cases where the surface is in contact with a melting solid or a boiling liquid (phase change). In these two

case, there is heat transfer across the surface while the temperature remains constant during this phase change.

b) Surface subjected to a heat flux (Neumann condition)

If, now, for the spatial position M_0 , a heat flux q is imposed, the boundary condition in this case becomes

$$-\lambda \frac{\partial T(M_0, t)}{\partial M} = q \quad (\text{II.12})$$

This boundary condition, also known as the second-type condition, arises during surface heating by a heat flux, such as with an electrical resistance.

If the spatial position M_0 is adiabatic, and thus isolated, the boundary condition is then

$$\frac{\partial T(M_0, t)}{\partial M} = 0 \quad (\text{II.13})$$

c) Surface in contact with a fluid (third-type condition) Finally, it may occur that for the spatial position M_0 , heat exchange takes place by convection; the boundary condition in this case is written

$$-\lambda \frac{\partial T(M_0, t)}{\partial M} = h(T(M_0, t) - T_\infty) \quad (\text{II.14})$$

II. 3. Another way to derive the heat equation

II. 3. 1. Cartesian coordinates

Consider a elementary cube with sides dx , dy , and dz surrounding a stationary medium at temperature $T(x, y, z, t)$. The quantities of heat per condition perpendicular to each surface of the cube at points x , y , and z are

$$Q_x = -\lambda \frac{\partial T}{\partial x} dydz, \quad Q_y = -\lambda \frac{\partial T}{\partial y} dxdz, \quad Q_z = -\lambda \frac{\partial T}{\partial z} dxdy \quad (\text{II.15})$$

The quantities of heat conducted through the surfaces opposite to the previous surfaces are obtained by a first-order Taylor expansion, namely

$$Q_{x+dx} = Q_x + \frac{\partial Q_x}{\partial x} dx \quad (\text{II.16})$$

$$Q_{y+dy} = Q_y + \frac{\partial Q_y}{\partial y} dy \quad (\text{II.17})$$

$$Q_{z+dz} = Q_z + \frac{\partial Q_z}{\partial z} dz \quad (\text{II.18})$$

If the medium contains a volumetric heat source q_s , the energy generated E_g within the cube's volume is then

$$E_g = q_s dxdydz \quad (\text{II.19})$$

The energy stored E_{St} in the volume of the cube is, by definition,

$$E_{St} = \rho C_P \frac{\partial T}{\partial t} dxdydz \quad (\text{II.20})$$

The energy balance over the volume of the cube, in accordance with the principle of conservation of energy, is

$$E_e + E_g - E_s = E_{St} \quad (\text{II.21})$$

let

$$Q_x + Q_y + Q_z + q_s dxdydz - Q_{x+dx} - Q_{y+dy} - Q_{z+dz} = \rho C_P \frac{\partial T}{\partial t} dxdydz$$

using relations (II.16, 17, 18) we arrive at

$$\frac{\partial Q_x}{\partial x} dx - \frac{\partial Q_y}{\partial y} dy - \frac{\partial Q_z}{\partial z} dz + q_s dx dy dz = \rho C_p \frac{\partial T}{\partial t} dx dy dz \quad (\text{II.22})$$

By substituting the relations (II.15), we obtain

$$\begin{aligned} \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right) dx dy dz + \frac{\partial}{\partial y} \left(\lambda \frac{\partial T}{\partial y} \right) dx dy dz + \frac{\partial}{\partial z} \left(\lambda \frac{\partial T}{\partial z} \right) dx dy dz \\ + q_s dx dy dz = \rho C_p \frac{\partial T(x, y, z, t)}{\partial t} dx dy dz \end{aligned} \quad (\text{II.23})$$

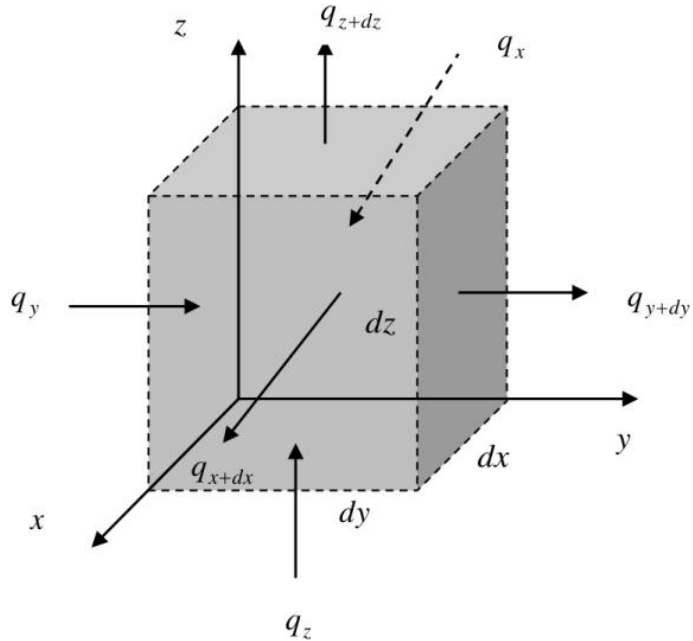


Figure II.2. Volume element $dv = dx dy dz$ in Cartesian coordinates.

by multiplying by the volume element $dx dy dz$, leading to the heat equation in Cartesian coordinates in its most general form

$$\frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(\lambda \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(\lambda \frac{\partial T}{\partial z} \right) + q_s = \rho C_p \frac{\partial T}{\partial t} \quad (\text{II.24})$$

If the thermal conductivity λ is constant, we recover equation (II.9), namely

$$\frac{\partial T}{\partial t} = a \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \frac{q_s}{\rho C_p} \quad (\text{II.25})$$

II. 3. 2. Cylindrical Coordinates

When the nabla operator ∇ is expressed in cylindrical coordinates, Fourier's law is written as

$$\vec{q} = -\lambda \left(\frac{\partial T(r, \varphi, z)}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial T(r, \varphi, z)}{\partial \varphi} \vec{e}_\theta + \frac{\partial T(r, \varphi, z)}{\partial z} \vec{e}_z \right) \quad (\text{II.26})$$

where

$$q_r = -\lambda \frac{\partial T}{\partial r}, \quad q_\varphi = -\frac{\lambda}{r} \frac{\partial T}{\partial \varphi}, \quad q_z = -\lambda \frac{\partial T}{\partial z} \quad (\text{II.27})$$

are respectively the components of the heat flux in the radial, circumferential, and axial directions.

By applying the principle of energy conservation to the volume element $rdrd\varphi dz$, the following heat equation is obtained

$$\rho C_p \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(\lambda r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \varphi} \left(\lambda \frac{\partial T}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left(\lambda \frac{\partial T}{\partial z} \right) + q_s \quad (\text{II.28})$$

If the thermal conductivity λ is constant, the equation becomes

$$\frac{\partial T}{\partial t} = a \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2} \right] + \frac{q_s}{\rho C_p} \quad (\text{II.29})$$

II. 2. 4. Spherical Coordinates

When the nabla operator $\vec{\nabla}$ is expressed in spherical coordinates, Fourier's law is written as

$$\vec{q} = -\lambda \left[\frac{\partial T(r, \varphi, \theta)}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial T(r, \varphi, \theta)}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial T(r, \varphi, \theta)}{\partial \varphi} \vec{e}_\varphi \right] \quad (\text{II.30})$$

where

$$q_r = -\lambda \frac{\partial T}{\partial r}, \quad q_\theta = -\frac{\lambda}{r} \frac{\partial T}{\partial \theta}, \quad q_\varphi = -\frac{\lambda}{r \sin \theta} \frac{\partial T}{\partial \varphi} \quad (\text{II.31})$$

are respectively the components of the heat flux in the radial, polar, and azimuthal directions. By applying the principle of energy conservation to the volume element $r^2 \sin \theta dr d\theta d\varphi$, the following heat equation is obtained

$$\begin{aligned} \rho C_p \frac{\partial T}{\partial t} = & \frac{1}{r^2} \frac{\partial}{\partial r} \left(\lambda r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \varphi} \left(\lambda \frac{\partial T}{\partial \varphi} \right) \\ & + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\lambda \sin \theta \frac{\partial T}{\partial \theta} \right) + q_s \end{aligned} \quad (\text{II.32})$$

If the thermal conductivity λ is constant, the equation becomes

$$\begin{aligned} \frac{\partial T}{\partial t} = & \alpha \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \varphi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) \right] \\ & + \frac{q_s}{\rho C_p} \end{aligned} \quad (\text{II.33})$$

II. 4. Special Forms of the Heat Equation

Let us return to the heat equation in its general form for the case where thermal conductivity is constant, namely

$$\frac{\partial T(M, t)}{\partial t} = a \Delta T(M, z) + \frac{q_s(M, t)}{\rho C_p} \quad (\text{II.34})$$

and let us consider the various particular cases.

II. 4. 1. Fourier's Equation

In the absence of internal heat sources ($q_s(M, t) = 0$), the heat equation becomes

$$\frac{\partial T(M, t)}{\partial t} = a \Delta T(M, z) \quad (\text{II.35})$$

this equation is called the Fourier equation.

II. 4. 2. Poisson Equation

In steady state, the temperature does not depend on time ($\frac{\partial T(M, t)}{\partial t} = 0$); the heat equation is known as the Poisson equation and is written

$$\Delta T(M) + \frac{q_s(M)}{\lambda} = 0 \quad (\text{II.36})$$

II. 4. 3. Laplace Equation

The Laplace equation is obtained in the case of steady-state regime and in the absence of internal heat sources ($q_s(M, t) = 0$). In this case, we obtain

$$\Delta T(M) = 0 \quad (\text{II.37})$$

II. 5. Heat Conduction in the Case of Simple Geometries under Steady-State Conditions and Without Internal Heat Sources

II. 5. 1. Flat wall

Consider a plane wall with thermal conductivity λ and thickness L in the Ox direction and infinite in the other two directions. To find the temperature distribution within this wall, one must then solve Laplace's equation in the Ox direction, namely

$$\frac{d^2T(x)}{dx^2} = 0 \quad (\text{II.38})$$

the boundary conditions in this case are

$$T(0) = T_1 \quad (\text{II.39})$$

$$T(L) = T_2 \quad (\text{II.40})$$

Integration of the heat equation yields

$$T(x) = C_1x + C_2 \quad (\text{II.41})$$

by applying the boundary conditions II. 39 and II. 40, we obtain the relations

$$T_1 = C_1 \times 0 + C_2 \quad (\text{II.42})$$

$$T_2 = C_1L + C_2 \quad (\text{II.43})$$

from which

$$C_2 = T_1 \quad (\text{II.44})$$

$$C_1 = \frac{T_2 - T_1}{L} \quad (\text{II.45})$$

by introducing these expressions into Equation II. 41, we obtain the temperature distribution in the wall

$$T(x) = (T_2 - T_1) \frac{x}{L} + T_1 \quad (\text{II.46})$$

or in a dimensionless form

$$\frac{T(x) - T_1}{T_2 - T_1} = \frac{x}{L} \quad (\text{II.47})$$

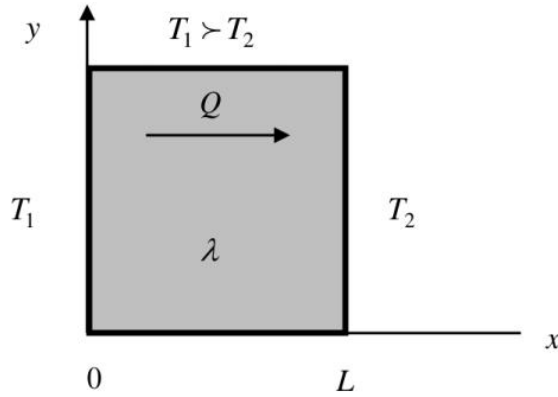


Figure II.3. Heat conduction in a plane wall.

The amount of heat passing through the wall is

$$Q = -\lambda S \frac{dT(x)}{dx} = \lambda S \frac{(T_1 - T_2)}{L} \quad (\text{II.48})$$

let us write this equation differently

$$Q = \frac{(T_1 - T_2)}{\frac{L}{\lambda S}} \quad (\text{II.49})$$

To determine the temperature distribution in a plane wall, one can start from the expression of the heat flux and not from the heat equation. Indeed, we have

$$Q = -\lambda S \frac{dT(x)}{dx} \quad (\text{II.50})$$

by integrating this equation as follows

$$Q \int_0^L dx = -\lambda S \int_{T_1}^{T_2} dT \quad (\text{II.51})$$

the amount of heat is therefore

$$Q = \frac{(T_1 - T_2)}{\frac{L}{\lambda S}} \quad (\text{II.52})$$

To find the temperature distribution, it suffices to write

que $Q \int_0^x dx = -\lambda S \int_{T_1}^{T(x)} dT$, soit

$$QL = S(T_1 - T(x)) \quad (\text{II.53})$$

by substituting Q with its expression (Equation II.52), one obtains the temperature distribution in the wall

$$\frac{T(x) - T_1}{T_2 - T_1} = \frac{x}{L} \quad (\text{II.54})$$

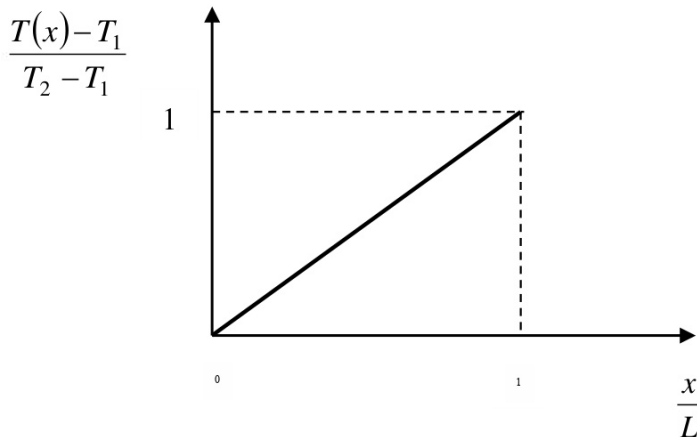


Figure II. 4. Temperature distribution in a plane wall.

II. 5. 2. Hollow Cylinder

Let a hollow cylinder have an inner radius r_i , an outer radius r_e , a length L , and be made of a material with thermal conductivity λ . Let T_1 and T_2 be the temperatures of the inner and outer walls of the cylinder, respectively. By radial symmetry, the temperature within the cylinder depends only on the radius r .

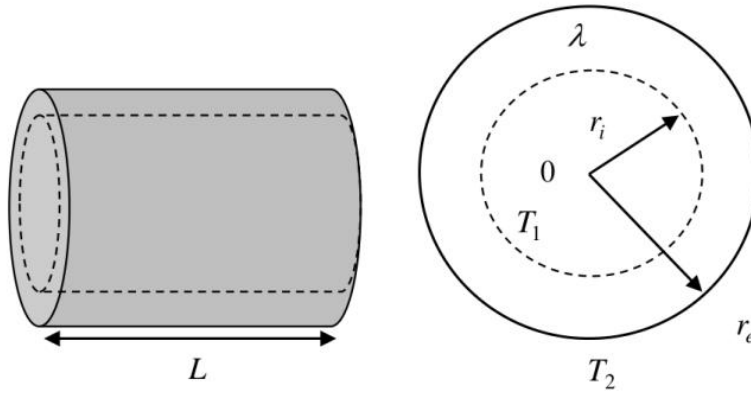


Figure II.5. Heat conduction in a hollow cylinder.

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad (\text{II.55})$$

Since heat conduction is radial, the temperature depends only on the radius r , and Laplace's equation reduces to the following differential equation

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT(r)}{dr} \right) = 0 \quad (\text{II.56})$$

with the boundary conditions

$$T(r_i) = T_1 \quad (\text{II.57})$$

$$T(r_e) = T_2 \quad (\text{II.58})$$

a first integration of this equation yields

$$r \frac{dT(r)}{dr} = C_1 \quad (\text{II.59})$$

hence

$$\frac{dT(r)}{dr} = \frac{C_1}{r} \quad (\text{II.60})$$

by integrating a second time, we obtain the temperature distribution in the cylinder

$$T(r) = C_1 \ln r + C_2 \quad (\text{II.61})$$

by calculating the integration constants, the temperature distribution in the cylinder is

$$\frac{T(r) - T_1}{T_2 - T_1} = \frac{\ln \frac{r}{r_i}}{\ln \frac{r_e}{r_i}} \quad (\text{II.62})$$

The heat flux passing through the cylinder is therefore

$$Q = -2\pi\lambda rL \frac{dT(r)}{dr} \quad (\text{II.63})$$

By differentiating the expression for the temperature and inserting its expression into the flux equation, one obtains

$$Q = \frac{T_1 - T_2}{\frac{1}{2\pi\lambda L} \ln \frac{r_e}{r_i}} \quad (\text{II.64})$$

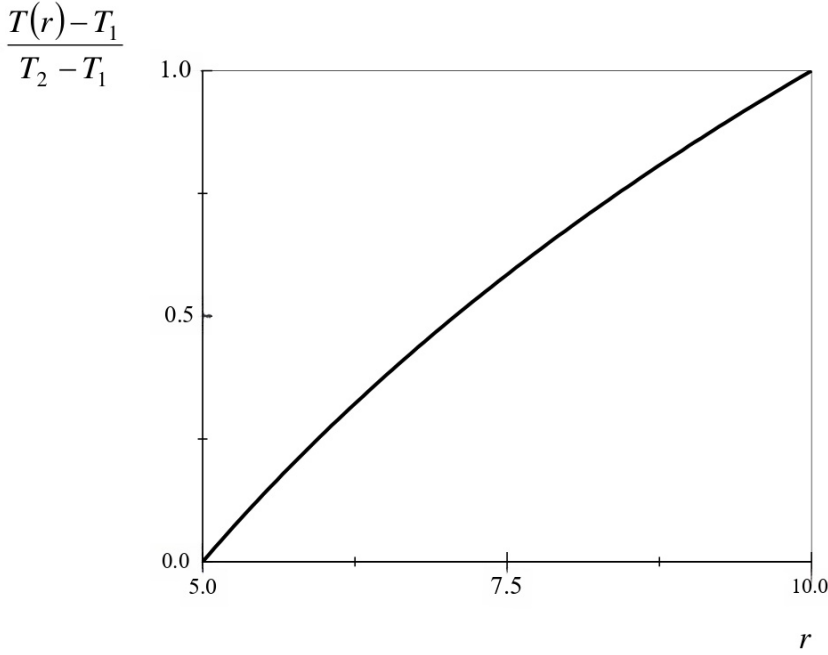


Figure II.6. Temperature profile in a hollow cylinder.

II. 5. 3. Hollow Sphere

Let a hollow sphere with inner radius r_i , outer radius r_e and composed of a material with thermal conductivity λ . Let T_1 and T_2 be the temperatures of the inner and outer walls of the sphere, respectively. The Laplace equation is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \varphi} \left(\frac{\partial T}{\partial \varphi} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) \quad (\text{II.65})$$

Since heat conduction is radial, the temperature depends only on the radius r , and the Laplace equation reduces to the following differential equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dT(r)}{dr} \right) = 0 \quad (\text{II.66})$$

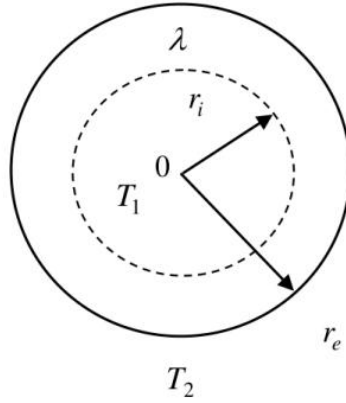


Figure II. 7. Heat conduction in a hollow sphere.

The boundary conditions are

$$T(r_i) = T_1 \quad (\text{II.67})$$

$$T(r_e) = T_2 \quad (\text{II.68})$$

a first integration yields

$$r^2 \frac{dT(r)}{dr} = C_1 \quad (\text{II.69})$$

hence

$$\frac{dT(r)}{dr} = \frac{C_1}{r^2} \quad (\text{II.70})$$

by integrating a second time, we obtain the temperature distribution

Heat Transfer I

$$T(r) = -\frac{C_1}{r} + C_2 \quad (\text{II.71})$$

by calculating the integration constants, the temperature across the sphere becomes

$$\frac{T(r) - T_1}{T_2 - T_1} = \frac{\frac{1}{r} - \frac{1}{r_i}}{\frac{1}{r_e} - \frac{1}{r_i}} \quad (\text{II.72})$$

The heat flux passing through the sphere

$$Q = -4\pi\lambda r^2 \frac{dT(r)}{dr} \quad (\text{II.73})$$

by differentiating the expression for the temperature and inserting its expression into the flux equation, we obtain

$$Q = \frac{T_1 - T_2}{\frac{r_e - r_i}{4\pi\lambda r_e r_i}} \quad (\text{II.74})$$

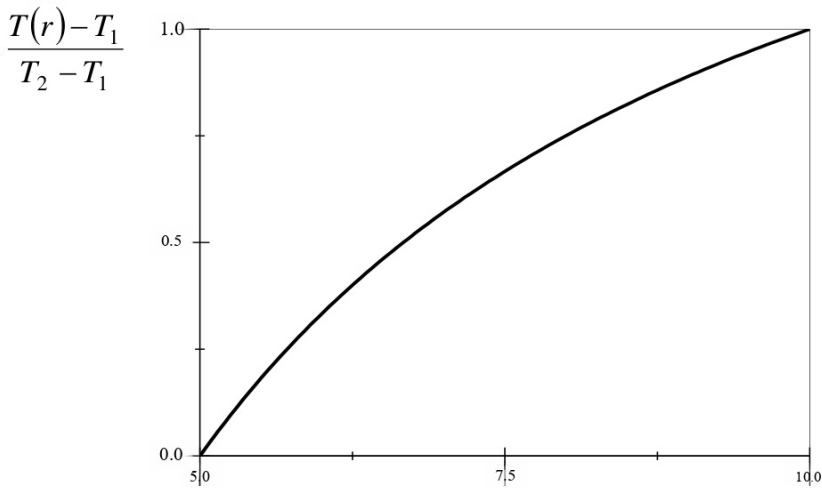


Figure II.8. Temperature profile in a hollow sphere.

II. 6. Heat conduction in the case of simple geometries under steady-state conditions and with heat sources

II.6.1. Plane wall

II. 6. 1. Wall plan

Consider a plane wall with thermal conductivity λ and thickness L in the Ox direction, infinite in the two other directions where there is a constant internal heat source q_s .

To determine the temperature distribution in this wall, one must then solve the Poisson equation, namely

$$\frac{d^2 T(x)}{dx^2} + \frac{q_s}{\lambda} = 0 \quad (\text{II.75})$$

with the boundary conditions

$$T(0) = T_1 \quad (\text{II.76})$$

$$T(L) = T_2 \quad (\text{II.77})$$

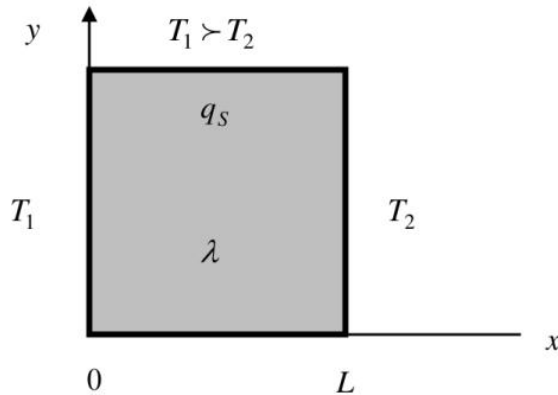


Figure II.9. Heat conduction in a plane wall.

$$T(x) = -\frac{q_s}{2\lambda} x^2 + C_1 x + C_2 \quad (\text{II.78})$$

where C_1 and C_2 are integration constants to be determined from the boundary conditions.

The boundary conditions provide the following relations for C_1 and C_2

$$T_1 = -\frac{q_s}{2\lambda} \times 0^2 + C_1 \times 0 + C_2 \quad (\text{II.79})$$

$$T_2 = -\frac{q_s}{2\lambda} L^2 + C_1 L + C_2 \quad (\text{II.80})$$

hence

$$C_1 = \frac{T_2 - T_1}{L} + \frac{q_s}{2\lambda} L \quad (\text{II.81})$$

$$C_2 = T_1 \quad (\text{II.82})$$

by substituting the integration constants into Equation II.78, we obtain the temperature distribution in the wall

$$T(x) - T_1 = \frac{q_s L^2}{2\lambda} \left(\frac{x}{L} - \left(\frac{x}{L} \right)^2 \right) + (T_2 - T_1) \frac{x}{L} \quad (\text{II.83})$$

If we assume that the two temperatures T_1 and T_2 are equal to T_0 , then the temperature distribution within the wall becomes

$$T(x) - T_0 = \frac{q_s L^2}{2\lambda} \left(\frac{x}{L} - \left(\frac{x}{L} \right)^2 \right) \quad (\text{II.84})$$

Let us find the position where the temperature is maximum within the wall; for this, let us calculate the derivative of Equation II. 84, namely

$$\frac{dT(x)}{dx} = \frac{q_s L^2}{2\lambda} \left(\frac{1}{L} - \frac{2x}{L^2} \right) \quad (\text{II.85})$$

the temperature is maximum if

$$\frac{dT(x)}{dx} = 0 \quad (\text{II.86})$$

that is, for

$$x = \frac{L}{2} \quad (\text{II.87})$$

and from this, the maximum temperature is

$$T_{\max} = \frac{q_s L^2}{8\lambda} + T_0 \quad (\text{II.88})$$

II. 6. 2. Full Cylinder

Consider a hollow solid cylinder with outer radius r_e , length L , composed of a material with thermal conductivity λ and subject to a constant heat generation rate q_s . In this case, the Poisson equation is written as

In this case, the Poisson equation is written as

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{q_s}{\lambda} = 0 \quad (\text{II.89})$$

if we multiply both sides of the equation by r , we obtain

$$\frac{d}{dr} \left(r \frac{dT}{dr} \right) = - \frac{q_s r}{\lambda} \quad (\text{II.90})$$

A first integration leads to the expression

$$r \frac{dT}{dr} = -\frac{q_s}{2\lambda} r^2 + C_1 \quad (\text{II.91})$$

by integrating once more, we obtain the temperature distribution, namely

$$T(r) = -\frac{q_s}{4\lambda} r^2 + C_1 \ln r + C_2 \quad (\text{II.92})$$

the integration constants are obtained from the following boundary conditions

$$T(r_e) = T_0 \quad (\text{II.93})$$

$$\frac{dT(0)}{dr} = 0 \quad (\text{II.94})$$

from which

$$C_1 = 0 \quad (\text{II.95})$$

$$C_2 = \frac{q_s}{4\lambda} r_e^2 + T_0 \quad (\text{II.96})$$

from these values, the temperature becomes

$$T(r) - T_0 = \frac{q_s r_e^2}{4\lambda} \left(1 - \left(\frac{r}{r_e} \right)^2 \right) \quad (\text{II.97})$$

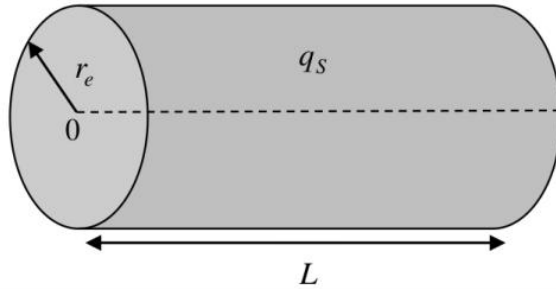


Figure II.10. Solid cylinder heat generation site.

II. 7. Analogy between Heat Transfer and Electricity In electricity, the potential difference $V_2 - V_1$ across the terminals of an electrical resistance R is proportional to the electric current I passing through it; this is Ohm's Law.

$$I = \frac{V_2 - V_1}{R} \quad (\text{II.98})$$

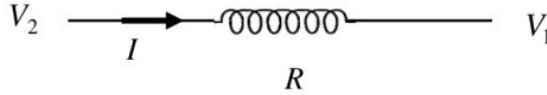


Figure II.11. Illustration of Ohm's Law.

by rewriting the equation

$$Q = \frac{(T_1 - T_2)}{\frac{L}{\lambda S}} \quad (\text{II.99})$$

and if we make a comparison between this equation and Ohm's law, we can correspond the quantity of heat Q to the electric current I and the temperature difference $T_1 - T_2$ to the potential difference $V_1 - V_2$. Thus, starting from relation II. 99, we obtain an electrical analogy for heat transfer by conduction, by defining a thermal resistance R given by

$$Q = \frac{(T_1 - T_2)}{\frac{L}{\lambda S}} = \frac{(T_1 - T_2)}{R} \quad (\text{II.100})$$

let

$$R = \frac{L}{\lambda S} \quad (\text{II.101})$$

In the same way, we define the thermal resistance for the cylinder by the relation

$$R = \frac{1}{2\pi\lambda L} \ln \frac{r_e}{r_i} \quad (\text{II.102})$$

and for the sphere via the relation

$$R = \frac{r_e - r_i}{4\pi\lambda r_e r_i} \quad (\text{II.103})$$

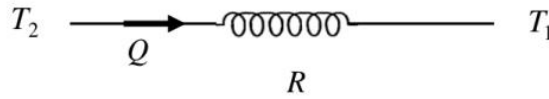


Figure II.12. Thermal circuit analogous to Ohm's law.

II. 8. Planar Walls in Series

In many practical cases, a flat wall may be composed of several different layers assembled in series. This situation is illustrated in Figure 6.

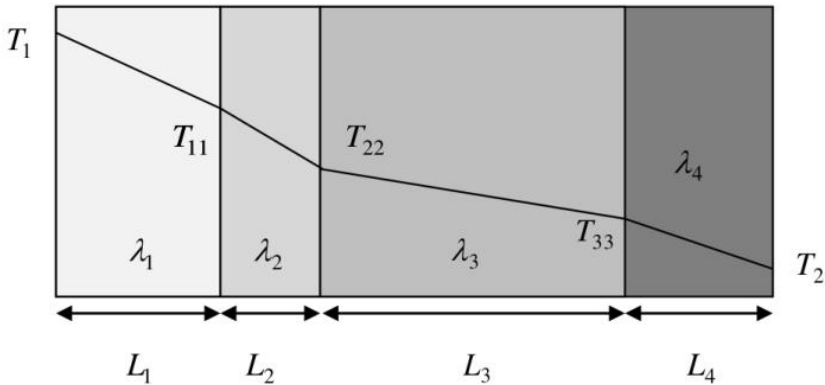


Figure II. 13. Planar wall formed by several layers in series.

The amount of heat passing through the first layer is

$$Q = \frac{T_1 - T_{11}}{R_1} \quad (\text{II.104})$$

the same amount of heat also passes through the second layer, the third layer, and so on. Therefore, by rewriting Equation II. 104 for the other layers, we find

$$Q = \frac{T_1 - T_{11}}{R_1} = \frac{T_{11} - T_{22}}{R_2} = \frac{T_{22} - T_{33}}{R_3} = \frac{T_{33} - T_2}{R_4} \quad (\text{II.105})$$

this relation can be written in the form

$$Q = \frac{T_1 - T_{11} + T_{11} - T_{22} + T_{22} - T_{33} + T_{33} - T_2}{R_1 + R_2 + R_3 + R_4} = \frac{T_1 - T_2}{R_1 + R_2 + R_3 + R_4}$$

Consequently, the amount of heat passing through the layers in series is

$$Q = \frac{T_1 - T_2}{R_1 + R_2 + R_3 + R_4} = \frac{T_1 - T_2}{R_{\acute{e}q}} \quad (\text{II.106})$$

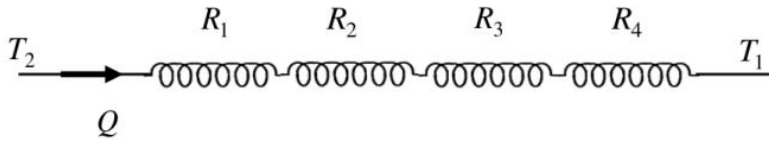


Figure II. 14. Analog thermal circuit for a wall composed in series.

where the equivalent thermal resistance is defined by the relation

$$R_{eq} = R_1 + R_2 + \dots R_n \quad (\text{II.107})$$

or else

$$R_{eq} = \sum_{n=1}^{n=N} \frac{L_n}{\lambda_n S} \quad (\text{II.108})$$

II. 9. Series-parallel planar walls

Unlike the first case where the wall is composed of several layers in series, the planar wall may also be composed of several layers arranged in parallel as shown in Figure 8.

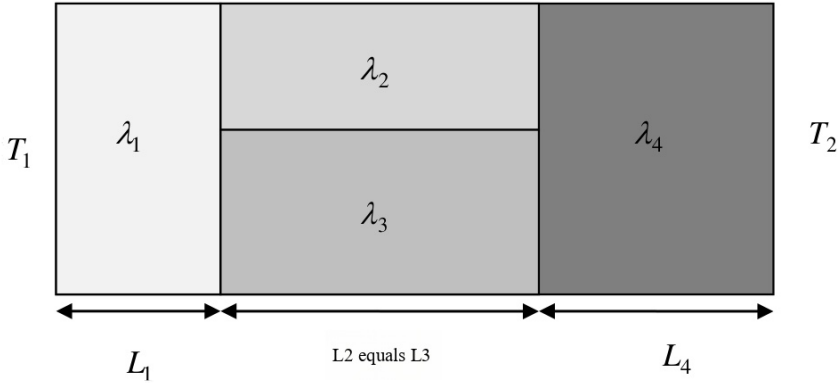


Figure II. 15. Planar wall formed by several series-parallel layers.

In this case, the equivalent thermal resistance is given by the expression

$$R_{eq} = R_1 + \frac{R_2 R_3}{R_2 + R_3} + R_4 \quad (\text{II.109})$$

where

$$R_1 = \frac{L_1}{\lambda_1 S_1}, R_2 = \frac{L_2}{\lambda_2 S_2}, R_3 = \frac{L_2}{\lambda_3 S_3}, R_4 = \frac{L_4}{\lambda_4 S_4} \quad (\text{II.110})$$

The individual resistances of each layer of the surface through which heat transfer occurs must be taken into account in the calculation, i.e., the surface that is perpendicular to the heat flux.

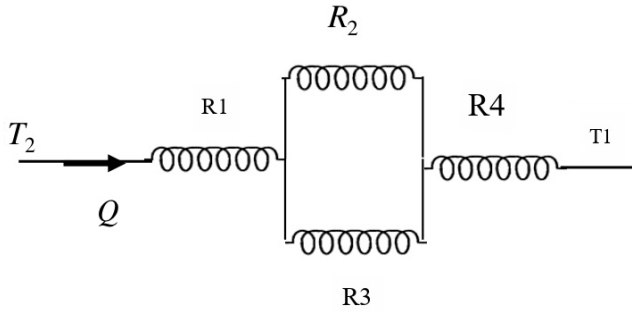


Figure II. 16. Analog thermal circuit for a wall composed in series-parallel.

II. 10. Combined Heat Transfer Mechanisms

In this typical case, there are three modes of heat transfer. A mode of heat transfer by convection between the walls and the ambient fluid, a mode of transfer by radiation between the walls and the ambient fluid, and the third mode by conduction through the solid layer. Let T_∞ denote the temperature of the ambient fluid, and let T_1 and T_2 denote the temperatures of the walls of the solid layer.

Assuming that heat transfer occurs from left to right ($T_\infty > T'_\infty$), the quantity of heat exchanged by convection is

$$Q_{conv1} = h_1 S (T_\infty - T_1) \quad (\text{II.111})$$

in terms of thermal resistance, we have

$$Q_{conv1} = \frac{T_\infty - T_1}{R_{conv1}} \quad (\text{II.112})$$

where the convective thermal resistance is defined by the relation

$$R_{conv1} = \frac{1}{h_1 S} \quad (\text{II.113})$$

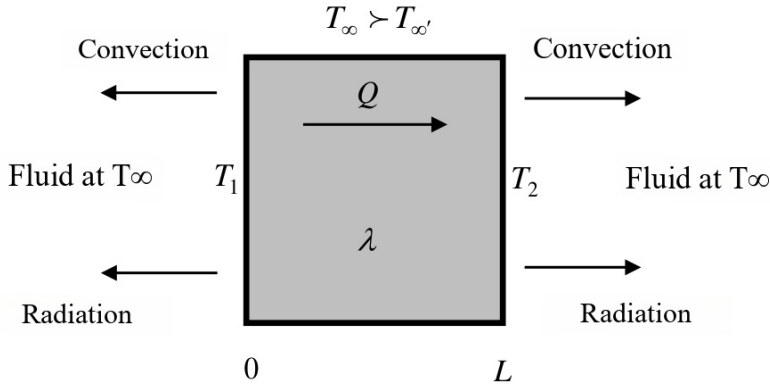


Figure II. 17. Combined heat transfer mechanism.

The heat exchanged by radiation is

$$Q_{rayo1} = \varepsilon_1 \sigma S (T_\infty^4 - T_1^4) \quad (\text{II.114})$$

this can be expressed in terms of thermal resistance to radiation in the form

$$q_{rayo1} = \frac{T_\infty - T_1}{R_{rayo1}} \quad (\text{II.115})$$

$$R_{rayo1} = \frac{T_\infty - T_1}{\varepsilon_1 \sigma S (T_\infty^4 - T_1^4)} \quad (\text{II.116})$$

the heat exchanged by conduction is

Heat Transfer I

$$Q_{cond} = \frac{(T_1 - T_2)}{\frac{L}{\lambda S}} \quad (\text{II.117})$$

the thermal conduction resistance is consequently

$$R_{cond} = \frac{L}{\lambda S} \quad (\text{II.118})$$

As on the right we have heat exchange by convection and by radiation, we can also write that

$$Q_{conv2} = h_2 S (T_2 - T_{\infty'}) \quad (\text{II.119})$$

$$R_{conv2} = \frac{1}{h_2 S} \quad (\text{II.120})$$

Similarly, for the heat exchanged by radiation, we have

$$q_{rayo2} = \varepsilon_2 \sigma S (T_2^4 - T_{\infty'}^4) \quad (\text{II.121})$$

where the thermal resistance due to radiation is

$$R_{rayo2} = \frac{T_2 - T_{\infty'}}{\varepsilon_2 \sigma S (T_2^4 - T_{\infty'}^4)} \quad (\text{II.122})$$

by combining these formulas, the exchanged heat is

$$Q = \frac{T_{\infty} - T_{\infty'}}{R_{eq}} = \frac{T_{\infty} - T_{\infty'}}{\frac{R_{conv1} R_{rayo1}}{R_{conv1} + R_{rayo1}} + R_{cond} + \frac{R_{conv2} R_{rayo2}}{R_{conv2} + R_{rayo2}}} \quad (\text{II.123})$$

It is often useful to define the overall heat transfer coefficient U by the equation

$$Q = US(T_{\infty} - T_{\infty'}) \quad (\text{II.124})$$

By comparing equations II. 123 and II. 124, the overall exchange coefficient with respect to surface S is therefore

$$US = \frac{1}{\frac{R_{conv1} R_{rayo1}}{R_{conv1} + R_{rayo1}} + R_{cond} + \frac{R_{conv2} R_{rayo2}}{R_{conv2} + R_{rayo2}}} \quad (\text{II.125})$$

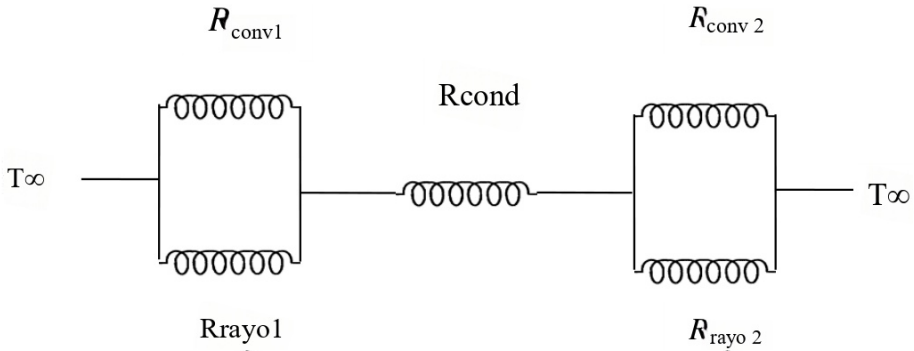


Figure II. 18. Equivalent electrical circuit.

II. 11. Critical thickness of thermal insulation for a conduit

We know that when we add to a wall, we increase the thermal resistance and consequently reduce the amount of heat exchanged. This is natural, as the exchange surface of a wall is constant; adding insulation always increases the thermal resistance of the wall without increasing the convective thermal resistance. However, in the case of a cylinder, things are very different. Adding insulation to a cylinder increases the conductive thermal resistance while simultaneously decreasing the convective thermal resistance. Consequently, the amount of heat exchanged may increase or decrease depending on which of the two effects is dominant.

Consider a cylindrical conduit with inner radius r_i and outer radius r_e . We cover this conduit with a thermal insulator of thickness e and thermal conductivity λ_{isol} . Let the temperature of the inner wall of the pipe be T_0 and the ambient fluid be at temperature T_∞ . If h denotes the convective heat transfer coefficient between the outer surface of the pipe and the ambient fluid, the thermal resistance by convection is therefore

$$R_{conv} = \frac{1}{2\pi(r_e + e)Lh} \quad (\text{II.126})$$

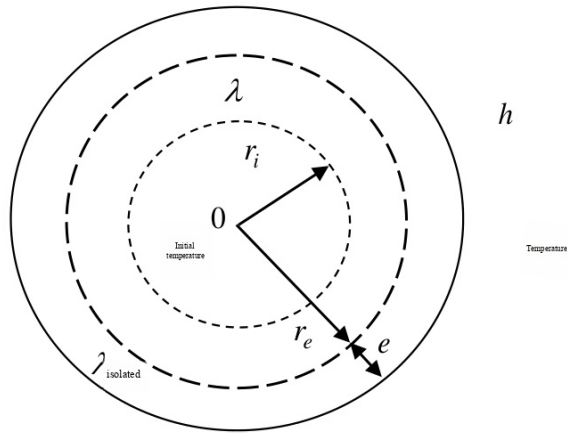


Figure II. 19. Critical insulation thickness of a cylindrical pipe.

The thermal resistances by conduction of the conduit wall and of the thermal insulating layer are

$$R_{cond1} = \frac{1}{2\pi\lambda L} \ln \frac{r_e}{r_i} \quad (\text{II.127})$$

Heat Transfer I

$$R_{cond\ 2} = \frac{1}{2\pi\lambda_{isol}L} \ln \frac{r_e + e}{r_e} \quad (II.128)$$

the amount of heat exchanged is therefore

$$Q = \frac{T_0 - T_\infty}{R_{conv} + R_{cond1} + R_{cond2}} \quad (II.129)$$

in terms of the conduction rays, this quantity of heat is

$$Q = \frac{T_0 - T_\infty}{\frac{1}{2\pi\lambda L} \ln \frac{r_e}{r_i} + \frac{1}{2\pi\lambda_{isol}L} \ln \frac{r_e + e}{r_e} + \frac{1}{2\pi(r_e + e)Lh}} \quad (II.130)$$

the equivalent thermal resistance is

$$R_e = \frac{1}{2\pi\lambda L} \ln \frac{r_e}{r_i} + \frac{1}{2\pi\lambda_{isol}L} \ln \frac{r_e + e}{r_e} + \frac{1}{2\pi(r_e + e)Lh} \quad (II.131)$$

Let us find the extremum of this expression by calculating the derivative with respect to the thickness of the thermal insulation; this extremum is determined from the relation

$$\frac{dR_e}{de} = 0 \quad (II.132)$$

let

$$\frac{dR_e}{de} = \frac{1}{2\pi\lambda_{isol}L} \left[\frac{\frac{1}{r_e}}{\frac{r_e + e}{r_e}} \right] - \frac{2\pi Lh}{(2\pi(r_e + e)Lh)^2} \quad (II.133)$$

or else

$$\frac{dR_e}{de} = \frac{1}{2\pi\lambda_{isol}L(r_e + e)} - \frac{2\pi Lh}{(2\pi(r_e + e)Lh)^2} = 0 \quad (\text{II.134})$$

that is to say

$$\frac{1}{2\pi\lambda_{isol}L(r_e + e)} = \frac{2\pi Lh}{4\pi^2(r_e + e)^2 L^2 h^2} \quad (\text{II.135})$$

from which

$$r_e + e_c = \frac{\lambda_{isol}}{h} \quad (\text{II.136})$$

Here e_c is the critical insulation thickness. Note that this critical thickness depends on the thermal conductivity of the insulator used and the convective heat transfer coefficient. We have plotted in Figures II. 20 (a) and II. 20 (b) the variations of the equivalent thermal resistance and the quantity of heat exchanged as a function of thickness.

We note that for a thickness e less than the critical thickness e_c , the thermal resistance decreases and consequently the amount of heat exchanged increases. The thermal resistance is minimal for $e = e_c$ and consequently the quantity of heat reached is maximum for this thickness. Beyond this critical thickness, adding insulation increases the thermal resistance and this leads to an inevitable decrease in the amount of heat exchanged. Thus, when insulating pipes or tanks, it is imperative to ensure that the thickness of the insulation used is greater than the critical thickness.

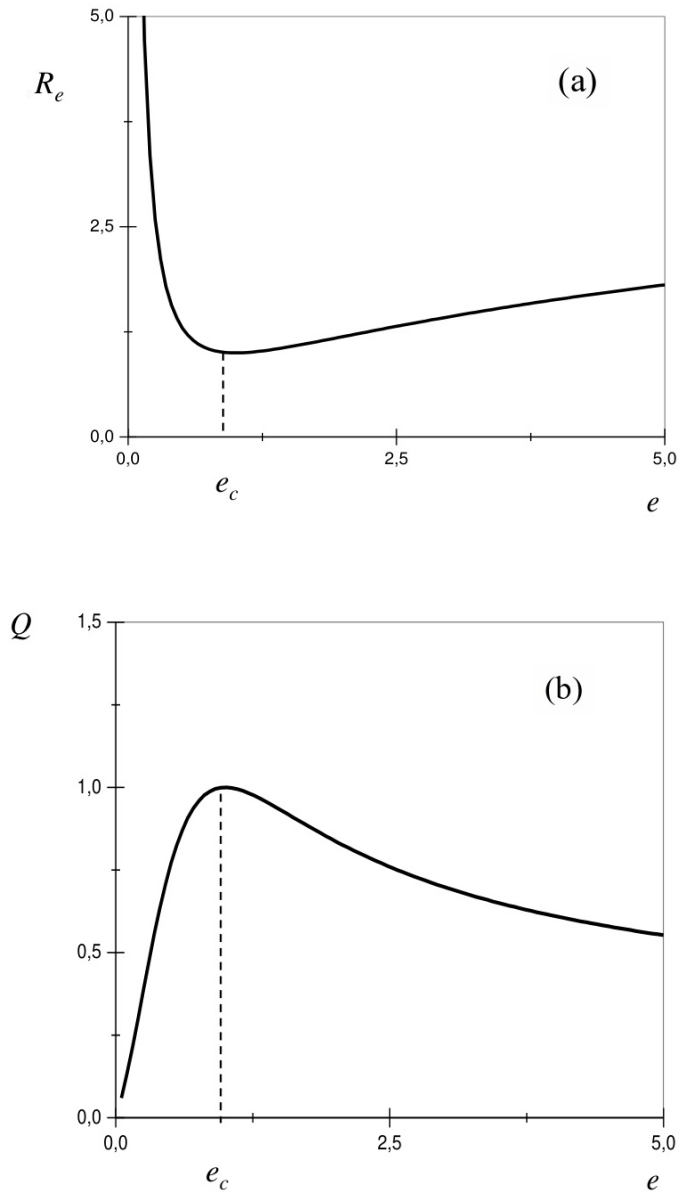


Figure II. 20. Variation of thermal resistance and heat quantity as a function of insulation thickness.

II. 12. Ailettes

Fins are small surfaces attached to a body immersed in a fluid. The purpose of fins is to increase the heat dissipation from the body to the fluid.

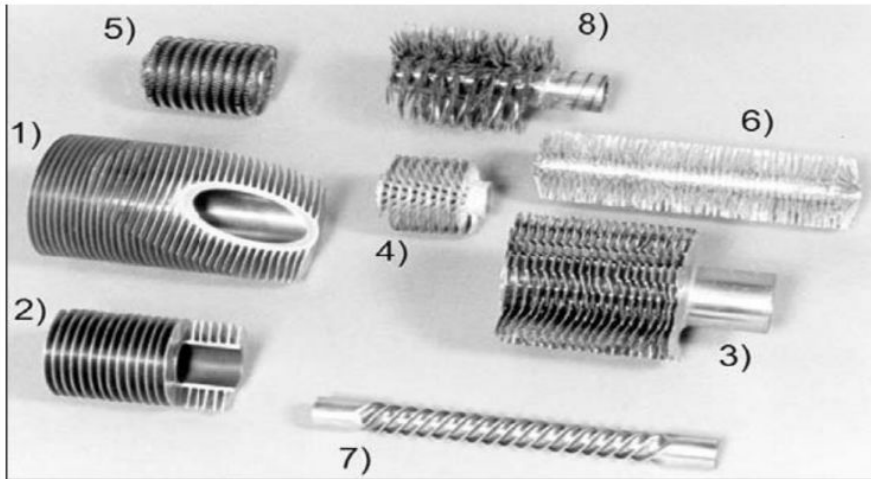


Figure II.21. Examples of fins.

For this, consider an infinitesimal element of a fin with cross-sectional area $S_s(x)$, thickness dx , and lateral surface area $dS_L(x)$ immersed in a fluid at temperature T_∞ . Let us write an energy balance for this infinitesimal element.

$$Q_x = Q_{x+dx} + dQ_{conv} \quad (\text{II.137})$$

From Fourier's law, we have

$$Q_x = -\lambda S_s(x) \frac{dT(x)}{dx} \quad (\text{II.138})$$

the amount of heat at the point $x + dx$ is

$$Q_{x+dx} = Q_x + \frac{dQ_x}{dx} dx \quad (\text{II.139})$$

using Equation II. 137, we find

$$Q_{x+dx} = -\lambda S_s(x) \frac{dT(x)}{dx} - \lambda \frac{d}{dx} \left(S_s(x) \frac{dT(x)}{dx} \right) dx \quad (\text{II.140})$$

the amount of heat exchanged by convection between the fin element and the fluid is

$$dQ_{conv} = h(T(x) - T_\infty) dS_L(x) \quad (\text{II.141})$$

By substituting relations II. 140 and II. 141 into the energy balance, we arrive at the equation

$$\begin{aligned} -\lambda S_s(x) \frac{dT(x)}{dx} &= -\lambda S_s(x) \frac{dT(x)}{dx} - \lambda \frac{d}{dx} \left(S_s(x) \frac{dT(x)}{dx} \right) dx \\ &+ h(T(x) - T_\infty) dS_L(x) \end{aligned} \quad (\text{II.142})$$

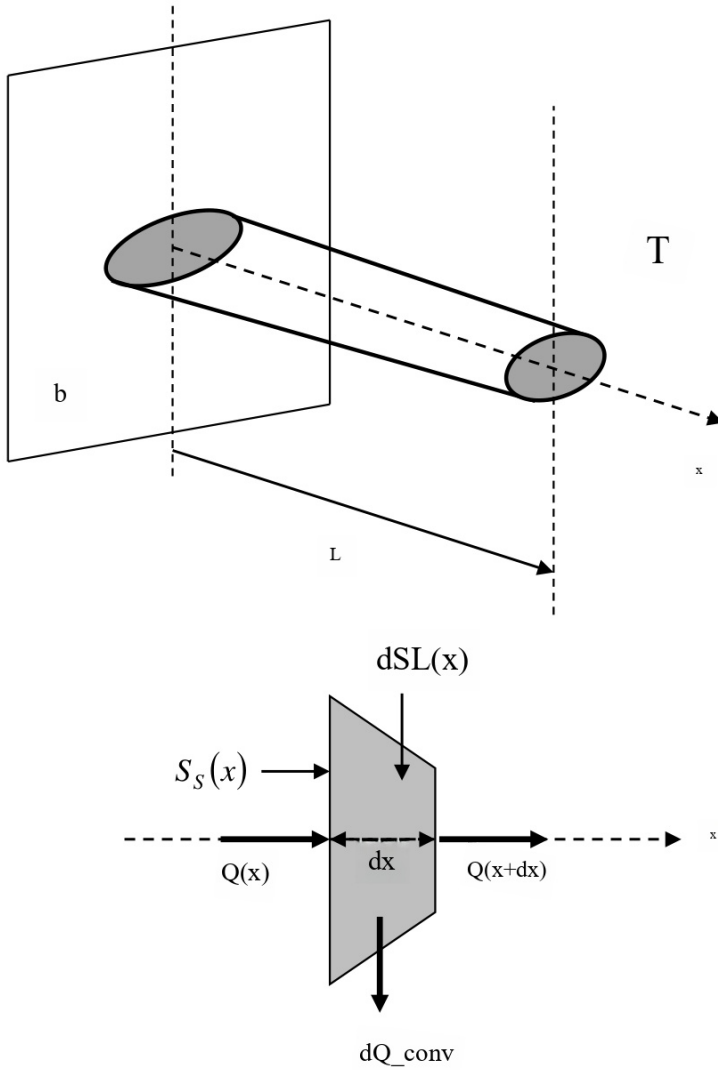


Figure II. 22. Illustration of heat transfer in a fin.
let

$$\frac{d}{dx} \left(S_s(x) \frac{dT(x)}{dx} \right) - \frac{h}{\lambda} (T(x) - T_\infty) \frac{dS_L(x)}{dx} = 0 \quad (\text{II.143})$$

or else

$$\frac{d^2 T(x)}{dx^2} + \left(\frac{1}{S_S(x)} \frac{dS_S(x)}{dx} \right) \frac{dT(x)}{dx} - \left(\frac{1}{S_S(x)} \frac{h}{\lambda} \frac{dS_L(x)}{dx} \right) (T(x) - T_\infty) = 0 \quad (\text{II.144})$$

This is the general equation for a fin. Its solution, subject to appropriate boundary conditions, yields the temperature distribution within the fin and, consequently, the amount of heat exchanged between the fin and the surrounding fluid.

II. 12. 1. Uniform cross-section fins

Let us begin with the case of a straight, rectangular fin of uniform cross-section S . The fin is fixed at its base to a surface at a constant temperature T_b and extends into a fluid medium at a constant temperature T_∞ .

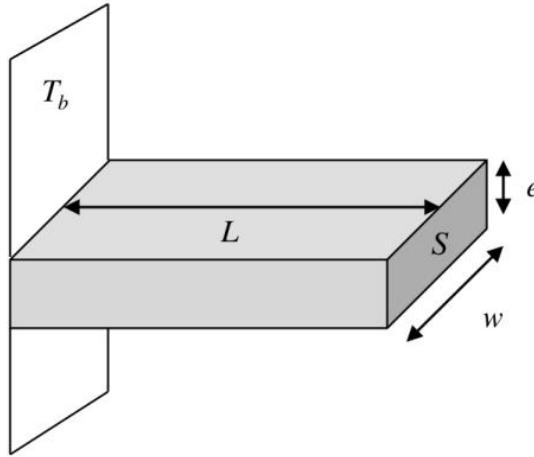


Figure II. 23. Uniform-section airfoil.

In this case, the surfaces $S_S(x)$ and $S_L(x)$ are

$$S_S(x) = S \quad (\text{II.145})$$

$$S_L(x) = Px \quad (\text{II.146})$$

where P is the perimeter of the fin cross-section, consequently

$$\frac{dS_s(x)}{dx} = 0 \quad (\text{II.147})$$

$$\frac{dS_L(x)}{dx} = P \quad (\text{II.148})$$

by substituting these expressions into the general equation of the bladelet, we obtain the equation

$$\frac{d^2 T(x)}{dx^2} - m^2 (T(x) - T_\infty) = 0 \quad (\text{II.149})$$

where $m = \sqrt{\frac{hP}{\lambda S}}$ is a constant parameter.

Physically, the boundary conditions for this type of problem are

a) The temperature at the base of the fin is the surface temperature of the body to which this fin is attached; this provides us with the first boundary condition.

$$T(0) = T_b \quad (\text{II.150})$$

b) For the second boundary condition, there may be three possibilities, which are

1) The fin is very long; in this case, the fin temperature becomes equal to the fluid temperature

$$T(\infty) = T_\infty \quad (\text{II.151})$$

2) The tip of the fin can be thermally isolated, i.e.,

$$\left. \frac{dT(x)}{dx} \right|_{x=L} = 0 \quad (\text{II.152})$$

3) If the fin is not insulated at the end, it exchanges heat with the fluid surrounding it; in this case, the following boundary condition applies to the temperature

$$-\lambda \left. \frac{dT(x)}{dx} \right|_{x=L} = h(T(L) - T_{\infty}) \quad (\text{II.153})$$

The solution to Equation II. 149 is

$$T(x) - T_{\infty} = C_1 e^{mx} + C_2 e^{-mx} \quad (\text{II.154})$$

where C_1 and C_2 are integration constants determined from the boundary conditions. By applying the boundary conditions to the different thermal configurations cited above.

1) Very long finlet

$$\frac{T(x) - T_{\infty}}{T_b - T_{\infty}} = e^{-mx} \quad (\text{II.155})$$

2) Isolated fin at the end

$$\frac{T(x) - T_{\infty}}{T_b - T_{\infty}} = \frac{\cosh m(L - x)}{\cosh mL} \quad (\text{II.156})$$

3) The fin exchanges heat by convection with the fluid surrounding it

$$\frac{T(x) - T_{\infty}}{T_b - T_{\infty}} = \frac{\cosh m(L - x) + \frac{h}{m\lambda} \sinh m(L - x)}{\cosh mL + \frac{h}{m\lambda} \sinh mL} \quad (\text{II.157})$$

Heat Transfer I

The amount of heat exchanged by the fin is calculated in two different ways, namely starting from Fourier's law applied to the base of the fin

$$Q_a = -\lambda S \left(\frac{dT(x)}{dx} \right)_{x=0} \quad (\text{II.158})$$

either based on the heat removed by convection to the fluid through the lateral surface of the fin

$$Q_a = hP \int_0^L (T(x) - T_\infty) dx \quad (\text{II.159})$$

Calculate for each case the amount of heat dissipated by the fin using Fourier's law for simplicity.

For a very long fin, the amount of heat dissipated is

$$Q_a = \sqrt{\lambda h S P} (T_b - T_\infty) \quad (\text{II.160})$$

For an isolated fin at the end, the amount of heat dissipated is

$$Q_a = \sqrt{\lambda h S P} (T_b - T_\infty) \tanh mL \quad (\text{II.161})$$

For the case where there is convective exchange between the tip of the fin and the surrounding medium, the amount of heat dissipated is

$$Q_a = \sqrt{\lambda h S P} (T_b - T_\infty) \frac{\sinh mL + \frac{h}{m\lambda} \cosh mL}{\cosh mL + \frac{h}{m\lambda} \sinh mL} \quad (\text{II.162})$$

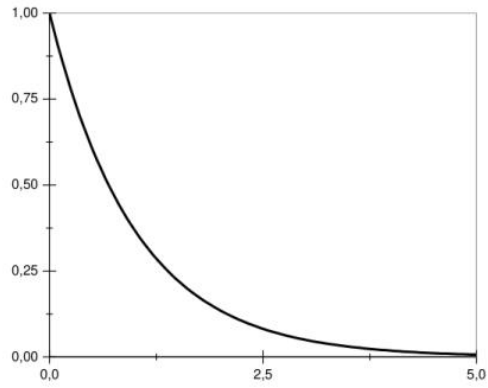


Figure II. 24. Temperature profile for a very long fin.

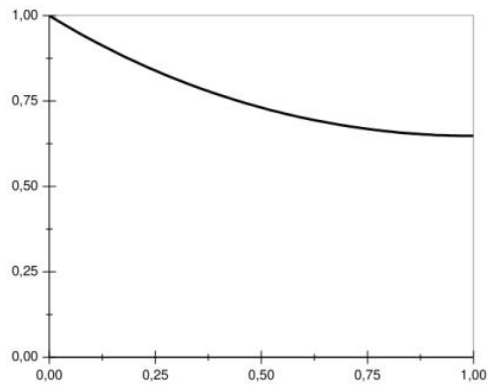


Figure II. 25. Temperature profile for an isolated fin at the tip.

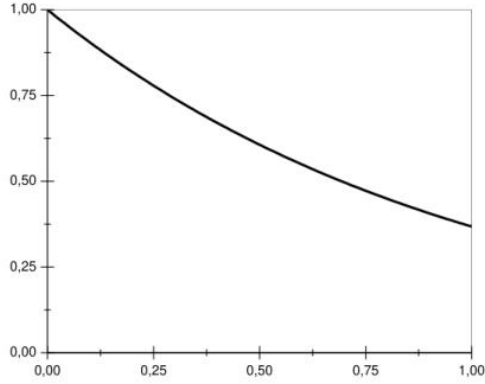


Figure II. 26. Temperature profile for a fin exchanging heat with the surrounding medium by convection.

II. 12. 2. Slat Efficiency

To characterize the fins, we define the efficiency η_a as the ratio of the heat actually dissipated by the fin to the maximum heat that could be dissipated by the fin if its temperature were equal to the base temperature T_b . The quantity of maximum heat dissipated by the fin is therefore

$$Q_{\max} = hPL(T_b - T_{\infty}) \quad (\text{II.163})$$

the efficiency η_a of the fin is thus given by the expression

$$\eta_a = \frac{Q_a}{Q_{\max}} = \frac{-\lambda S \left(\frac{dT(x)}{dx} \right)_{x=0}}{hPL(T_b - T_{\infty})} \quad (\text{II.164})$$

By combining equations II. 160, II. 161 and II. 162 with equation II. 164, we obtain for the studied cases the following expressions for the yields

a) Very long wing

$$\eta_a = \frac{1}{mL} \quad (\text{II.165})$$

b) End plate

$$\eta_a = \frac{\tanh mL}{mL} \quad (\text{II.166})$$

c) The fin exchanges heat by convection with the fluid surrounding it

$$\eta_a = \frac{1}{mL} \frac{\sinh mL + \frac{h}{m\lambda} \cosh mL}{\cosh mL + \frac{h}{m\lambda} \sinh mL} \quad (\text{II.167})$$

II. 12. 3. Fin efficiency

Fins are used to enhance heat transfer, and the addition of fins to a surface to improve heat transfer must necessarily justify the cost and complexity associated with this operation. There is no guarantee that adding fins increases heat transfer. Fin efficiency is based on the increase in heat transfer relative to that exchanged by a surface without fins. The efficiency ε_a of a fin is defined by the relation

$$\varepsilon_a = \frac{Q_a}{Q_{\text{sans ailette}}} = \frac{Q_a}{hS_b(T_b - T_\infty)} \quad (\text{II.168})$$

Let us note that the efficiency η_a of the fin and its effectiveness ε_a are intimately related. Indeed, we have

$$\varepsilon_a = \frac{\eta_a h S_a (T_b - T_\infty)}{h S_b (T_b - T_\infty)} \quad (\text{II.169})$$

let

$$\varepsilon_a = \eta_a \frac{S_a}{S_b} \quad (\text{II.170})$$

Thus, the efficiency ε_a can be determined from the fin efficiency η_a and vice versa.

For a very long airfoil with uniform cross-section, and since $S = S_b$, the efficiency ε_a is given by

$$\varepsilon_a = \frac{\sqrt{\lambda h S P} (T_b - T_\infty)}{h S_b (T_b - T_\infty)} = \sqrt{\frac{\lambda P}{h S}} \quad (\text{II.171})$$

From this relation, we can draw the following conclusions:

- a) Thermal conductivity should be as high as possible. This is why the fins are made of aluminum, a material with high thermal conductivity and exceptional lightness, as well as resistance to corrosion compared to other materials.
- b) The ratio $\frac{P}{S}$ should be as large as possible. This criterion is satisfied by thin fins.
- c) The use of fins is even more effective when the convective heat transfer coefficient is low. This situation occurs when the external medium is a gas and convection is natural.

II. 12. 3. Efficiency of a Finned Surface

To determine the amount of heat exchanged by a finned surface, one must take into account the unfinned surface S_{sa} as well as the fin surface S_a . In this case, the amount of heat exchanged by the finned surface is

$$Q = Q_{sa} + Q_a \quad (\text{II.172})$$

let

$$Q = hS_{sa}(T_b - T_\infty) + \eta_a hS_a(T_b - T_\infty) \quad (\text{II.173})$$

or else

$$Q = h(S_{sa} + \eta_a S_a)(T_b - T_\infty) \quad (\text{II.174})$$

we can now define the overall efficiency η_g for a finned surface by the relation

$$\eta_g = \frac{Q}{Q_b} = \frac{h(S_{sa} + \eta_a S_a)(T_b - T_\infty)}{hS_b(T_b - T_\infty)} \quad (\text{II.175})$$

let

$$\eta_g = \frac{S_{sa} + \eta_a S_a}{S_b} \quad (\text{II.176})$$