

Chapter 1 — Introduction to Mathematical Logic

This chapter introduces the basic language of mathematics and, especially, how to write mathematical proofs correctly.

1 Axiom, Lemma, Theorem, and Conjecture

These four words are very important.

1.1 Axiom

Definition 1.1 (Axiom). An **axiom** is a statement that we accept as true without proving it within a mathematical system.

Example 1.1. Through two distinct points, there is exactly one straight line.

We do not prove this from other results; it is taken as a starting point in a particular geometric system.

Think of an axiom as a basic rule of the game.

1.2 Lemma

Definition 1.2 (Lemma). A **lemma** is a mathematical statement that has been proved and is mainly used as a tool to prove another, usually more important, result.

Example 1.2. If a and b are even integers, then $a + b$ is even.

Proof. Since a and b are even, there exist integers m, n such that

$$a = 2m, \quad b = 2n.$$

Therefore,

$$a + b = 2m + 2n = 2(m + n).$$

Since $m + n$ is an integer, $a + b$ is even. □

This lemma can then be used in a larger theorem.

1.3 Theorem

Definition 1.3 (Theorem). A **theorem** is an important mathematical statement that has been proved rigorously.

Example 1.3. The sum of two even integers is even.

Its proof can be exactly the one we gave above.

A theorem usually has the form:

$$\boxed{\text{Hypotheses} \implies \text{Conclusion}}$$

For example:

If n is even, then n^2 is even.

Here:

- Hypothesis: n is even.
- Conclusion: n^2 is even.

1.4 Conjecture

Definition 1.4 (Conjecture). A **conjecture** is a statement that is believed to be true but has not yet been proved.

Example 1.4. Every even integer greater than 2 can be written as the sum of two prime numbers.

This is the famous **Goldbach conjecture**.

It has been tested for extremely large ranges of numbers, but numerical testing is not the same as a mathematical proof.

1.5 Important distinction

Term	Meaning
Axiom	Accepted without proof
Lemma	Proven result used as a tool
Theorem	Proven mathematical result
Conjecture	Believed to be true but not proven

2 Mathematical Proof

Definition 2.1 (Proof). A **mathematical proof** is a logical sequence of statements showing that a conclusion follows from accepted facts, definitions, axioms, and previously proven results.

A proof should be:

- logical,
- clear,

- complete,
- precise.

Proposition 2.1. *If n is odd, then n^2 is odd.*

Proof. Since n is odd, there exists an integer k such that

$$n = 2k + 1.$$

Then

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Since $2k^2 + 2k$ is an integer, n^2 has the form $2m + 1$.

Therefore, n^2 is odd. □

Notice the structure:

Definition $\rightarrow$ Calculation $\rightarrow$ Conclusion

3 How to Write a Mathematical Proof

A very useful structure is:

Step 1 — Identify what is given. For example:

Let n be an even integer.

Step 2 — Use the definition. An even integer can be written as:

$$n = 2k, \quad k \in \mathbb{Z}.$$

Step 3 — Perform the necessary reasoning. For example:

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Step 4 — State the conclusion. Since $2k^2 \in \mathbb{Z}$,

$$n^2 = 2(2k^2),$$

so n^2 is even.

4 “Without Loss of Generality” — WLOG

The expression “without loss of generality” (WLOG) means that we can consider one particular case because the other cases are essentially equivalent by symmetry.

Example 4.1. Suppose we want to prove:

If a and b are even integers, then $a + b$ is even.

There is no need to distinguish between a and b . We can simply write:

$$a = 2m, \quad b = 2n.$$

But WLOG becomes particularly useful when the problem has symmetric variables.

For example:

Let $a, b \in \mathbb{R}$. Suppose $a \leq b$. WLOG, we can consider the ordering $a \leq b$.

The important point is:

Warning: You cannot use WLOG whenever you want. You must be able to explain why the other case is equivalent.

For example, if a statement is symmetric in a and b , exchanging a and b does not change the problem.

5 Constructive Proof

Definition 5.1 (Constructive Proof). A **constructive proof** proves that an object exists by actually constructing it or giving a method for finding it.

Example 5.1. Statement: There exists an even prime number.

A constructive proof is very simple: take

$$p = 2.$$

The number 2 is prime and even.

Therefore, an even prime number exists.

$$\boxed{2}$$

This is constructive because we explicitly produced the object.

Example 5.2. Statement: There exists an integer n such that

$$n^2 = 9.$$

Constructively, choose

$$n = 3.$$

Then

$$3^2 = 9.$$

Therefore, such an integer exists.

6 Existential Proof

Definition 6.1 (Existential Proof). An **existential proof** proves a statement of the form:

$$\exists x, P(x)$$

which means:

“There exists an x such that $P(x)$ is true.”

Sometimes we provide an explicit example, and sometimes we prove existence indirectly.

Example 6.1. Prove:

$$\exists x \in \mathbb{R}, \quad x^2 = 2.$$

We know that

$$x = \sqrt{2}$$

is a real number and

$$(\sqrt{2})^2 = 2.$$

Therefore,

$$\boxed{\exists x \in \mathbb{R}, \quad x^2 = 2.}$$

7 Important Proof Techniques to Know

7.1 Direct proof

To prove

$$P \Rightarrow Q,$$

assume P and logically derive Q .

Example 7.1. Prove:

If n is divisible by 4, then n is even.

Since $4 \mid n$, there exists $k \in \mathbb{Z}$ such that

$$n = 4k.$$

Therefore,

$$n = 2(2k).$$

Thus n is even.

7.2 Proof by Contradiction

To prove a statement P :

1. Assume that P is false.
2. Derive a contradiction.
3. Conclude that P must be true.

Example 7.2. Prove that $\sqrt{2}$ is irrational.

Suppose, for contradiction, that

$$\sqrt{2} = \frac{p}{q}$$

where $p, q \in \mathbb{Z}$, $q \neq 0$, and the fraction is in lowest terms.

Then

$$2 = \frac{p^2}{q^2}$$

so

$$p^2 = 2q^2.$$

Thus p^2 is even, so p is even.

Let

$$p = 2k.$$

Then

$$4k^2 = 2q^2$$

and therefore

$$q^2 = 2k^2.$$

Thus q is also even.

So both p and q are even, contradicting the fact that p/q was in lowest terms.

Therefore,

$$\boxed{\sqrt{2} \notin \mathbb{Q}.$$

8 Exercises

Try solving these without looking at the solutions first.

Exercise 8.1 (Definitions). Classify each statement as axiom, theorem, lemma, or conjecture.

1. A statement accepted as a starting point without proof.
2. A proven result used mainly to establish another result.
3. A mathematical statement that has been rigorously proved.
4. A statement believed to be true but not yet proved.

Exercise 8.2 (Even numbers). Prove:

If n is even, then n^2 is even.

Hint: Start with

$$n = 2k.$$

Exercise 8.3 (Odd numbers). Prove:

If n is odd, then n^2 is odd.

Hint:

$$n = 2k + 1.$$

Exercise 8.4 (Existence). Prove constructively that:

$$\exists n \in \mathbb{Z}, \quad n^2 = 25.$$

Exercise 8.5 (Existence). Prove that there exists an integer that is both even and a multiple of 3.

Try to give an explicit example.

Exercise 8.6 (Constructive). Prove that there exist two irrational numbers x and y such that xy is rational.

Exercise 8.7 (Direct proof). - Prove:

If a and b are odd integers, then $a + b$ is even.

Hint:

$$a = 2m + 1, \quad b = 2n + 1.$$

- Let $a, b \in \mathbb{R}^+$, Show that if $a \leq b$ then $a \leq \frac{a+b}{2} \leq b$ and $a \leq \sqrt{ab} \leq b$.

Exercise 8.8. Prove:

If n is an integer and n^2 is even, then n is even.

Hint: You can use proof by contradiction.