

Tutorial Exercises 1

Exercise 1 – Estimating a Transition Matrix from Data

The load of a computer server is recorded once per minute, giving the following sequence of 13 observations:

$L, L, M, H, M, M, L, L, M, H, H, M, L$

where

$L = \text{Low}, \quad M = \text{Medium}, \quad H = \text{High}.$

Assume that the load follows a Markov chain.

1. Define the state space E .
2. For each state, count the transitions to every state and present the counts in a table.
3. Build the transition matrix $\hat{P}$.

Exercise 2 – Representing a Markov Chain as a Transition Graph

A website has three pages,

$$E = \{A, B, C\}.$$

A user navigates as follows:

- From A , the user goes to B or to C , with probability 0.5 each.
 - From B , the user always goes to C .
 - From C , the user always goes back to A .
1. Write the transition matrix P .
 2. Draw the directed transition graph of the chain.
 3. Compute P^2 .
 4. Starting from page A , what is the probability of being on page A after two clicks?
 5. List the possible two-step paths from A back to A and give the probability of each path.
 6. Verify your answer to Question 4 using the Chapman–Kolmogorov equation.

Exercise 3 – Stationary Distribution

Let (X_n) be a Markov chain with two states, 1 and 2, such that

$$p_{12} = 0.25, \quad p_{21} = 0.5.$$

1. Write the transition matrix P and draw the corresponding state-transition graph.
2. Calculate P^3 and deduce the transition probability from state 1 to state 2 in three steps,

$$p_{12}^{(3)}.$$

3. Verify your answer to Question 2 using the Chapman–Kolmogorov equation.
4. Given the initial distribution

$$\pi_0 = (0.2, 0.8),$$

Calculate the distributions

$$\pi_1, \quad \pi_2, \quad \pi_3,$$

5. Determine the stationary distribution π .

Exercise 4 – Communicating Classes, Recurrence, and Periodicity

Consider the Markov chain with transition matrix

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0.2 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

1. Draw the transition graph.
2. Determine which states communicate with each other.
3. Identify all communicating classes.
4. Determine which communicating classes are closed.
5. Classify the states as recurrent or transient.
6. Determine the period of each class.
7. Find all stationary distributions of the chain.
8. Explain why the stationary distribution is concentrated on the closed recurrent class.

Exercise 5 – Modeling the Spread of a Disease

Consider a population in which individuals may contract a disease. An individual can be in one of the following three states: $E = \{\text{Susceptible}, \text{Sick}, \text{Immune}\}$. A **Susceptible** individual is healthy but not immune, a **Sick** individual has the disease, and an **Immune** individual is healthy and partially protected against the disease..

The state of an individual may change from one day to the next according to the following rules:

- A Susceptible individual remains Susceptible with probability 0.95 and becomes Sick with probability 0.05.
- A Sick individual remains Sick with probability 0.90 and becomes Immune with probability 0.10.
- An Immune individual remains Immune with probability 0.98 and becomes Susceptible with probability 0.02.

Assume that the evolution of an individual's health status can be modeled as a discrete-time Markov chain, with one time step corresponding to one day.

1. Construct the transition matrix P of the corresponding Markov chain where S, M, and I denote Susceptible, Sick, and Immune, respectively. Draw the corresponding directed transition graph.
2. Initially, 4% of the population is Sick and the remaining 96% is Susceptible. No individual is initially Immune.

Calculate the proportion of the population in each state after 30 days.

To facilitate the calculations, use the following approximation:

$$P^{15} = \begin{pmatrix} 0.487 & 0.261 & 0.253 \\ 0.101 & 0.226 & 0.673 \\ 0.185 & 0.051 & 0.764 \end{pmatrix}.$$

3. Determine whether the Markov chain is **irreducible**, **aperiodic**, and **recurrent**. Justify each answer.
4. Determine whether the chain admits a **unique stationary distribution**. Justify your answer and compute the stationary distribution.
5. A computer calculation of the 100th power of the transition matrix gives the following result, rounded to three decimal places:

$$P^{100} = \begin{pmatrix} 0.250 & 0.125 & 0.625 \\ 0.250 & 0.125 & 0.625 \\ 0.250 & 0.125 & 0.625 \end{pmatrix}.$$

Suppose that the population consists of 50 million individuals. What will be the approximate number of individuals in each state in the long run?