

Chapter 2

Second-Order Linear Partial Differential Equations

2.1 Preliminaries

A partial differential equation is an equation of the form

$$F(x, y, \dots, u, u'_x, u'_y, \dots, u''_{xx}, u''_{yy}, u''_{xy}, \dots) = 0, \quad (1)$$

where $u = u(x, y)$ is the unknown function of the real variables x, y to be determined, and F is a given function (in this chapter u and F are real functions); x, y are the independent variables, with

$$u'_x = \frac{\partial u}{\partial x}, \quad u'_y = \frac{\partial u}{\partial y}, \quad u''_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u''_{yy} = \frac{\partial^2 u}{\partial y^2}, \quad u''_{xy} = \frac{\partial^2 u}{\partial y \partial x}.$$

– Equation (1) is called *linear* if F is a linear function of the quantities $u, u'_x, u'_y, u''_{xx}, u''_{yy}, u''_{xy}, \dots$. For example, the second-order linear partial differential equation in two independent variables x and y is:

$$au''_{xx} + bu''_{xy} + cu''_{yy} + ku'_x + lu'_y + mu = f,$$

where a, b, c, k, l, m and f are functions of x and y .

Definition 36. The *order* of a partial differential equation is the order of the highest derivative appearing in the equation.

Remark 37. Consider

$$au''_{xx} + bu''_{xy} + cu''_{yy} + ku'_x + lu'_y + mu = f.$$

If a, b, c, k, l, m are given real numbers (a, b, c not all zero), the equation is a second-order linear partial differential equation with constant coefficients.

Example 38.

1. Laplace's equation:

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0$$

is a second-order linear partial differential equation.

2. $\frac{\partial^2 f(x, y)}{\partial x^2} = h(x, y)$ is a second-order linear partial differential equation.
3. The diffusion equation $\frac{\partial u(x, y)}{\partial x} - \frac{\partial^2 u(x, y)}{\partial y^2} = 0$ is a second-order linear partial differential equation.
4. The equation $\frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y} + \sin(u) = 0$ is a first-order nonlinear partial differential equation.
5. $\frac{\partial u(x, y)}{\partial x} + u(x, y) \frac{\partial^2 u(x, y)}{\partial y^2} = 0$ is a second-order nonlinear partial differential equation.
6. $\cos(xy^2) \frac{\partial u(x, y)}{\partial x} - y^2 \frac{\partial u(x, y)}{\partial y} = \tan(x^2 + y^2)$ is a first-order linear PDE.

2.2 Quasilinear Partial Differential Equations

Let $u(x, y)$ be a real function of two real variables defined on an open set Ω of $\mathbb{R}^2$.

Definition 39. A second-order quasilinear partial differential equation is an equation of the form

$$a(x, y) \frac{\partial^2 u(x, y)}{\partial x^2} + 2b(x, y) \frac{\partial^2 u(x, y)}{\partial x \partial y} + c(x, y) \frac{\partial^2 u(x, y)}{\partial y^2} = F(x, y, u, p, q), \quad (\text{E})$$

where the unknown is the function $u(x, y)$, $p = \frac{\partial u(x, y)}{\partial x}$, $q = \frac{\partial u(x, y)}{\partial y}$, a, b, c are given functions on a domain Ω , and F is a real function.

2.2.1 Characteristic curves

Let $\gamma : x = \varphi(t), y = \psi(t)$ be a curve in $\mathbb{R}^2$.

Definition 40.

1. The characteristic curves of equation (E) are the curves γ satisfying:

$$\Delta(t) = c(\varphi(t), \psi(t)) [\varphi'(t)]^2 - 2b(\varphi(t), \psi(t)) \varphi'(t) \psi'(t) + a(\varphi(t), \psi(t)) [\psi'(t)]^2 = 0.$$

2. We say that γ is characteristic at no point if $\Delta(t) \neq 0$ for all t .

Theorem 41.

1. If $a \neq 0$, the characteristic curves are the solutions of the differential equation

$$a(x, y) \left(\frac{dy}{dx} \right)^2 - 2b(x, y) \left(\frac{dy}{dx} \right) + c(x, y) = 0.$$

2. If $c \neq 0$, they are the solutions of the differential equation

$$c(x, y) \left(\frac{dx}{dy} \right)^2 - 2b(x, y) \left(\frac{dx}{dy} \right) + a(x, y) = 0.$$

3. If $a = c = 0$, they are the lines $x = \text{constant}$ and $y = \text{constant}$.

2.2.2 Classification of equations

Consider equation (E) with $a(x, y) = a$, $b(x, y) = b$ and $c(x, y) = c$.

Definition 42.

- An equation such that $b^2 - ac > 0$ in a domain D is called *hyperbolic* in this domain; it admits two families of characteristic curves in D .
- If $b^2 - ac = 0$, equation (E) is called *parabolic* in D ; it admits only one family of characteristic curves in D .
- If $b^2 - ac < 0$, equation (E) is called *elliptic* in D ; it admits no (real) characteristic curves.

Example 43.

1.

$$\frac{\partial^2 u(x, y)}{\partial x^2} - \frac{\partial^2 u(x, y)}{\partial y^2} = F\left(x, y, u, \frac{\partial u(x, y)}{\partial x}, \frac{\partial u(x, y)}{\partial y}\right).$$

Here $b^2 - ac = 1$, so the equation is hyperbolic in $\mathbb{R}^2$; its characteristic curves are the solutions of $\left(\frac{dy}{dx}\right)^2 - 1 = 0$, namely the lines $y = x + k_1$ and $y = -x + k_2$.

2.

$$y^2 \frac{\partial^2 u(x, y)}{\partial x^2} - x^2 \frac{\partial^2 u(x, y)}{\partial y^2} = F\left(x, y, u, \frac{\partial u(x, y)}{\partial x}, \frac{\partial u(x, y)}{\partial y}\right).$$

We have $b^2 - ac = x^2 y^2$, so the equation is hyperbolic ($x \neq 0$, $y \neq 0$); the characteristic curves are the solutions of $y^2 \left(\frac{dy}{dx}\right)^2 - x^2 = 0$. Hence there are two families of curves:

$$y^2 - x^2 = k_1 \quad \text{and} \quad y^2 + x^2 = k_2, \quad k_1, k_2 \in \mathbb{R}.$$

3.

$$\frac{\partial^2 u(x, y)}{\partial x^2} + 2 \frac{\partial^2 u(x, y)}{\partial x \partial y} + \frac{\partial^2 u(x, y)}{\partial y^2} = F\left(x, y, u, \frac{\partial u(x, y)}{\partial x}, \frac{\partial u(x, y)}{\partial y}\right);$$

the characteristic curves are the lines $y = x + k$.

2.3 The Standard Form

Let $X = \varphi(x, y)$, $Y = \psi(x, y)$ be two new variables. We assume that

$$J(x, y) = \begin{vmatrix} \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} \end{vmatrix} \neq 0, \quad \forall x, y \text{ in an open set } \Omega.$$

Consider equation (E), where u is a quantity depending on x and y , with $u(x, y) = u(X, Y)$. Then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial X} \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial x}, \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial X} \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial y}, \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial X^2} \left(\frac{\partial X}{\partial x} \right)^2 + 2 \frac{\partial^2 u}{\partial X \partial Y} \left(\frac{\partial X}{\partial x} \right) \left(\frac{\partial Y}{\partial x} \right) + \frac{\partial^2 u}{\partial Y^2} \left(\frac{\partial Y}{\partial x} \right)^2 + \frac{\partial u}{\partial X} \frac{\partial^2 X}{\partial x^2} + \frac{\partial u}{\partial Y} \frac{\partial^2 Y}{\partial x^2}, \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial X^2} \left(\frac{\partial X}{\partial y} \right)^2 + 2 \frac{\partial^2 u}{\partial X \partial Y} \left(\frac{\partial X}{\partial y} \right) \left(\frac{\partial Y}{\partial y} \right) + \frac{\partial^2 u}{\partial Y^2} \left(\frac{\partial Y}{\partial y} \right)^2 + \frac{\partial u}{\partial X} \frac{\partial^2 X}{\partial y^2} + \frac{\partial u}{\partial Y} \frac{\partial^2 Y}{\partial y^2}, \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 u}{\partial X^2} \left(\frac{\partial X}{\partial x} \frac{\partial X}{\partial y} \right) + \frac{\partial^2 u}{\partial X \partial Y} \left(\frac{\partial X}{\partial x} \frac{\partial Y}{\partial y} + \frac{\partial X}{\partial y} \frac{\partial Y}{\partial x} \right) + \frac{\partial^2 u}{\partial Y^2} \left(\frac{\partial Y}{\partial x} \frac{\partial Y}{\partial y} \right) \\ &\quad + \frac{\partial u}{\partial X} \frac{\partial^2 X}{\partial x \partial y} + \frac{\partial u}{\partial Y} \frac{\partial^2 Y}{\partial x \partial y}.\end{aligned}$$

The terms containing second derivatives with respect to X and Y that express

$$a \frac{\partial^2 u(x, y)}{\partial x^2} + 2b \frac{\partial^2 u(x, y)}{\partial x \partial y} + c \frac{\partial^2 u(x, y)}{\partial y^2}$$

are

$$A \frac{\partial^2 u(X, Y)}{\partial X^2} + 2B \frac{\partial^2 u(X, Y)}{\partial X \partial Y} + C \frac{\partial^2 u(X, Y)}{\partial Y^2},$$

with

$$\begin{aligned}A &= a \left(\frac{\partial X}{\partial x} \right)^2 + 2b \frac{\partial X}{\partial x} \frac{\partial X}{\partial y} + c \left(\frac{\partial X}{\partial y} \right)^2, \\ B &= a \left(\frac{\partial X}{\partial x} \frac{\partial Y}{\partial x} \right) + b \left(\frac{\partial X}{\partial x} \frac{\partial Y}{\partial y} + \frac{\partial X}{\partial y} \frac{\partial Y}{\partial x} \right) + c \left(\frac{\partial X}{\partial y} \frac{\partial Y}{\partial y} \right), \\ C &= a \left(\frac{\partial Y}{\partial x} \right)^2 + 2b \left(\frac{\partial Y}{\partial x} \frac{\partial Y}{\partial y} \right) + c \left(\frac{\partial Y}{\partial y} \right)^2.\end{aligned}$$

We therefore have

$$B^2 - AC = J^2 (b^2 - ac),$$

which shows that the type of the equation is preserved under the change of variables.

– Hence, X and Y can be chosen so as to make at least one of the terms A , B and C vanish.

2.3.1 Hyperbolic equations

Theorem 44. *Let $\varphi_1(x, y) = k_1$ and $\varphi_2(x, y) = k_2$ be the two families of characteristic curves of a hyperbolic equation. Setting*

$$\begin{aligned}X_1 &= \varphi_1(x, y), \\ X_2 &= \varphi_2(x, y),\end{aligned}$$

the hyperbolic equation becomes

$$\frac{\partial^2 u}{\partial X_1 \partial X_2} = G \left(\frac{\partial u}{\partial X_1}, \frac{\partial u}{\partial X_2}, u, X_1, X_2 \right).$$

Setting

$$\begin{aligned} Y_1 &= X_1 + X_2, \\ Y_2 &= X_1 - X_2, \end{aligned}$$

it becomes

$$\frac{\partial^2 u}{\partial Y_1^2} - \frac{\partial^2 u}{\partial Y_2^2} = H\left(\frac{\partial u}{\partial Y_1}, \frac{\partial u}{\partial Y_2}, u, Y_1, Y_2\right).$$

Example 45. Consider

$$y^2 \frac{\partial^2 u(x, y)}{\partial x^2} - x^2 \frac{\partial^2 u(x, y)}{\partial y^2} = 0. \quad (\text{E})$$

The characteristic curves are $y^2 - x^2 = k_1$ and $y^2 + x^2 = k_2$. Let $x_1 = y^2 - x^2$ and $x_2 = y^2 + x^2$. Equation (E) becomes

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{x_1}{2(x_2^2 - x_1^2)} \frac{\partial u}{\partial x_2} - \frac{x_2}{2(x_2^2 - x_1^2)} \frac{\partial u}{\partial x_1}. \quad (\text{E}')$$

Now let $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$. Equation (E') becomes

$$\frac{\partial^2 u}{\partial y_1^2} - \frac{\partial^2 u}{\partial y_2^2} = \frac{1}{2y_2} \frac{\partial u}{\partial y_2} - \frac{1}{2y_1} \frac{\partial u}{\partial y_1}.$$

2.3.2 Parabolic equations

Theorem 46. Let $\varphi(x, y) = k$ be the family of characteristic curves of a parabolic equation, let $x_1 = \varphi(x, y)$ and let x_2 be a function independent of x_1 ($J \neq 0$). With these new variables, (E) becomes

$$\frac{\partial^2 u}{\partial x_2^2} = G\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, u, x_1, x_2\right).$$

Example 47. Consider the equation

$$x^2 \frac{\partial^2 u(x, y)}{\partial x^2} + 2xy \frac{\partial^2 u(x, y)}{\partial x \partial y} + y^2 \frac{\partial^2 u(x, y)}{\partial y^2} = 0. \quad (\text{E})$$

The characteristic curve is $\frac{y}{x} = c$. Setting $x_1 = \frac{y}{x}$ and $x_2 = y$, equation (E) becomes

$$x_2^2 \frac{\partial^2 u}{\partial x_2^2} = 0, \quad \text{or} \quad \frac{\partial^2 u}{\partial x_2^2} = 0.$$

2.3.3 Elliptic equations

The equation of the characteristic curves is

$$a \left(\frac{dy}{dx} \right)^2 - 2b \left(\frac{dy}{dx} \right) + c = 0.$$

This differential equation has no real solutions; we therefore find, in the complex plane,

$$\frac{dy}{dx} = \frac{b}{a} \pm i \sqrt{\frac{ac - b^2}{a^2}},$$

solved in the form

$$\begin{aligned}\eta_1(x, y) + i\psi_1(x, y) &= k_1, & k_1 \in \mathbb{C}, \\ \eta_2(x, y) + i\psi_2(x, y) &= k_2, & k_2 \in \mathbb{C}.\end{aligned}$$

We set $x_1 = \eta_1(x, y) + i\psi_1(x, y)$, $x_2 = \eta_2(x, y) + i\psi_2(x, y)$, and then $x_1 = y_1 + iy_2$, $x_2 = y_1 - iy_2$. The mixed term (the coefficient of $\frac{\partial^2 u}{\partial x \partial y}$) then vanishes.

Theorem 48. *Let $\varphi_1(x, y) = k_1$, $\varphi_2(x, y) = k_2$ be the solutions of*

$$\frac{dy}{dx} = \frac{b}{a} \pm i\sqrt{\frac{ac - b^2}{a^2}}.$$

Set $y_1 + iy_2 = \varphi_1(x, y)$ and $y_1 - iy_2 = \varphi_2(x, y)$. Then (E) becomes

$$\frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2} = G\left(\frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}, u, y_1, y_2\right).$$

Example 49. Consider (E):

$$\frac{\partial^2 u(x, y)}{\partial x^2} + x^2 \frac{\partial^2 u(x, y)}{\partial y^2} = 0.$$

The characteristic curves are

$$\begin{cases} 2y - ix^2 = k_1, \\ 2y + ix^2 = k_2. \end{cases}$$

We therefore set $y_1 = 2y$ and $y_2 = -x^2$; (E) becomes

$$\frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2} - \frac{1}{2y_2} \frac{\partial u}{\partial y_2} = 0.$$

2.4 Linear Equations with Constant Coefficients

Let (E) be a linear equation with constant coefficients of the following form:

$$a \frac{\partial^2 u(x, y)}{\partial x^2} + 2b \frac{\partial^2 u(x, y)}{\partial x \partial y} + c \frac{\partial^2 u(x, y)}{\partial y^2} + d \frac{\partial u(x, y)}{\partial x} + e \frac{\partial u(x, y)}{\partial y} + f u(x, y) = F(x, y),$$

where a, b, c, d, e, f are constants.

Theorem 50. *Let (E) be a linear equation with constant coefficients. Then one can find constants λ_1 and λ_2 such that, by setting*

$$u(x_1, x_2) = \varphi(x_1, x_2) \exp(\lambda_1 x_1 + \lambda_2 x_2),$$

(E) becomes:

$$(a) \quad \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = k\varphi + F(x_1, x_2) \quad \text{or} \quad \frac{\partial^2 \varphi}{\partial x_1^2} - \frac{\partial^2 \varphi}{\partial x_2^2} = c\varphi + G(x_1, x_2), \quad \text{if (E) is hyperbolic;}$$

$$(b) \quad \frac{\partial^2 \varphi}{\partial x_1^2} = k\varphi + F(x_1, x_2), \quad \text{if (E) is parabolic;}$$

(c) $\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} = k\varphi + F(x_1, x_2)$, if (E) is elliptic.

Example 51. Consider (E):

$$4 \frac{\partial^2 u(x, y)}{\partial x^2} + 5 \frac{\partial^2 u(x, y)}{\partial x \partial y} + \frac{\partial^2 u(x, y)}{\partial y^2} + \frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y} = 2.$$

The characteristic equation is $4 \left(\frac{dy}{dx}\right)^2 - 5 \frac{dy}{dx} + 1 = 0$, that is, the solutions are $y - x = k_1$ and $y - x/4 = k_2$.

Setting $x_1 = y - x$, $x_2 = y - x/4$, the canonical form is

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{1}{3} \frac{\partial u}{\partial x_2} - \frac{8}{9}.$$

Let

$$u(x_1, x_2) = \varphi(x_1, x_2) \exp(\lambda_1 x_1 + \lambda_2 x_2).$$

Then

$$\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = -\lambda_2 \frac{\partial \varphi}{\partial x_1} - \left(\lambda_1 - \frac{1}{3}\right) \frac{\partial \varphi}{\partial x_2} - \left(\lambda_1 \lambda_2 - \frac{\lambda_2}{3}\right) \varphi - \frac{8}{9} \exp(-(\lambda_1 x_1 + \lambda_2 x_2)).$$

Choosing $\lambda_1 = \frac{1}{3}$, $\lambda_2 = 0$, (E) becomes

$$\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = -\frac{8}{9} \exp\left(\frac{-x_1}{3}\right).$$