

Chapter 1: First-Order Partial Differential Equations

Theory, First Integrals, and the Method of Characteristics

Lecture Notes – Comprehensive Edition

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1 Introduction to Partial Differential Equations

1.1 What is a Partial Differential Equation?

A **partial differential equation** (PDE) is an equation involving an unknown function of several independent variables together with its partial derivatives. If $u = u(x_1, \dots, x_n)$ is the unknown function, a PDE has the general form

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}, u_{x_1 x_1}, \dots) = 0.$$

Unlike an ordinary differential equation, the unknown here depends on *several* variables, so its variation must be described in several directions at once. This is the natural language for phenomena that vary continuously in space and/or time.

Definition 1.1 (Order and linearity). The **order** of a PDE is the order of its highest derivative. A first-order PDE $F(x, y, u, u_x, u_y) = 0$ is:

- **linear** if F is linear in u, u_x, u_y with coefficients depending only on x, y ;
- **quasi-linear** if F is linear in u_x, u_y only, with coefficients possibly depending on u too;
- **nonlinear** if F depends nonlinearly on u_x and/or u_y .

1.2 Why Study PDEs? Applications

- **Transport equation** $u_t + cu_x = 0$: propagation of a concentration/density carried by a flow of constant speed c .
- **Traffic flow (LWR model)** $\rho_t + (v(\rho)\rho)_x = 0$: a first-order quasi-linear conservation law predicting the formation of traffic jams.
- **Heat equation** $u_t = ku_{xx}$: diffusion of heat, also used in financial mathematics (Black–Scholes).
- **Wave equation** $u_{tt} = c^2 u_{xx}$: vibrating strings, sound and electromagnetic waves.
- **Laplace/Poisson equation** $\Delta u = f$: steady-state electrostatics, temperature distribution.
- **Eikonal equation** $|\nabla u| = 1$: wavefront propagation in geometric optics.
- **Hamilton–Jacobi equations** $u_t + H(x, \nabla u) = 0$: classical mechanics and optimal control.

2 Mathematical Preliminaries

2.1 Partial Derivatives and Schwarz’s Theorem

Let $f(x, y)$ be a real function of two real variables defined on a neighborhood of a point $(a, b) \in \mathbb{R}^2$. If $x \mapsto f(x, b)$ is differentiable at $x = a$, its derivative is denoted $f_x(a, b)$, the **partial derivative** of f with respect to x at (a, b) . If $f_x(x, y)$ exists at every point of a neighborhood of (a, b) , this defines a new function, and similarly for f_y . We write

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}.$$

Theorem 2.1 (Schwarz). *If at a point of its domain the mixed second derivatives f_{xy}'' and f_{yx}'' exist and are continuous, then they are equal:*

$$f_{xy}'' = f_{yx}''.$$

Example 2.2. For $z(x, y) = x^3 - xy + y^2$ one checks directly that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = -1$.

2.2 Chain Rule for Composite Functions

Let $F(x, y) = f(u, v)$ where $u = u(x, y)$ and $v = v(x, y)$, and set $u_0 = u(x_0, y_0)$, $v_0 = v(x_0, y_0)$. If u, v have partial derivatives at (x_0, y_0) and f has continuous partial derivatives near (u_0, v_0) , then F has partial derivatives at (x_0, y_0) given by

$$\begin{aligned} F_x(x_0, y_0) &= \frac{\partial f}{\partial u}(u_0, v_0) \frac{\partial u}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial v}(u_0, v_0) \frac{\partial v}{\partial x}(x_0, y_0), \\ F_y(x_0, y_0) &= \frac{\partial f}{\partial u}(u_0, v_0) \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial f}{\partial v}(u_0, v_0) \frac{\partial v}{\partial y}(x_0, y_0). \end{aligned}$$

Example 2.3. If $f(u, v) = u^2 - uv + v^2$ with $u = x + y$, $v = x - y$, the chain rule above gives $F_x = u + v$ and $F_y = u - v$; in this particular example both simplify further to $F_x = 2x$ and $F_y = 2y$.

2.3 Total Differential

Definition 2.4. If $u(x, y)$ has continuous partial derivatives, its **total differential** is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

Example 2.5. For $u(x, y) = xy^3$, $du = y^3 dx + 3xy^2 dy$.

This last notion is the key computational tool of the whole chapter: throughout, we will look for functions u whose total differential $du = A dx + B dy + C dz$ matches a prescribed proportionality, which is exactly how *first integrals* are found.

3 The System $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

We work in $\mathbb{R}^3$ with an orthonormal frame $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, and consider a variable vector field

$$V(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)).$$

We study the symbolic differential system

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)} \quad (S)$$

Definition 3.1. A **solution** of (S) is a curve γ whose tangent at every point (x, y, z) where $V \neq 0$ is parallel to $V(x, y, z)$.

Definition 3.2 (Parametrized curve). A parametrized curve of $\mathbb{R}^3$ is a map $t \in [a, b] \mapsto \gamma(t) = (\varphi(t), \psi(t), \eta(t)) \in \mathbb{R}^3$.

Theorem 3.3. Let γ be a curve with parametrization (φ, ψ, η) and let $T = \left(\frac{d\varphi}{dt}, \frac{d\psi}{dt}, \frac{d\eta}{dt} \right)$ be its tangent vector. Then γ is a solution of (S) if and only if there exists a scalar function $K(t)$ such that

$$\frac{d\varphi}{dt} = K(t)P(x, y, z), \quad \frac{d\psi}{dt} = K(t)Q(x, y, z), \quad \frac{d\eta}{dt} = K(t)R(x, y, z).$$

When P, Q, R are all nonzero this is equivalent to writing

$$\frac{d\varphi/dt}{P(x, y, z)} = \frac{d\psi/dt}{Q(x, y, z)} = \frac{d\eta/dt}{R(x, y, z)}.$$

Proof. By definition, γ solves (S) iff at each point its tangent vector T is parallel to $V = (P, Q, R)$, i.e. $T = K(t)V$ for some scalar $K(t)$ (possibly depending on the parametrization). Writing this equality component-wise gives exactly the three relations above; dividing them when $P, Q, R \neq 0$ eliminates $K(t)$ and yields the symmetric ratio form. ■ □

Example 3.4. The system $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{0}$ is solved via the parametric method: setting $\frac{dx}{y} = \frac{dy}{x} = ds$ gives $\dot{x} = y$, $\dot{y} = x$, hence $\ddot{x} = x$, so $x(s) = Ae^s + Be^{-s}$, $y(s) = Ae^s - Be^{-s}$ — a parametric representation of the hyperbola $x^2 - y^2 = 4AB$ (with $z = \text{const}$).

4 First Integrals of (S)

4.1 Definition and Characterization

Definition 4.1. A **first integral** of (S) is a non-constant C^1 function $u(x, y, z)$ such that, along every solution curve $\gamma(t) = (\varphi(t), \psi(t), \eta(t))$ of (S) , the composition $u(\varphi(t), \psi(t), \eta(t))$ is constant. (In $\mathbb{R}^2$, a first integral of $\frac{dx}{P} = \frac{dy}{Q}$ is a function $u(x, y)$ constant along solutions.)

Theorem 4.2. A C^1 function u is a first integral of (S) if and only if, on every domain where solutions of (S) are defined,

$$P(x, y, z) \frac{\partial u}{\partial x} + Q(x, y, z) \frac{\partial u}{\partial y} + R(x, y, z) \frac{\partial u}{\partial z} = 0.$$

Proof. Let (φ, ψ, η) parametrize a solution γ of (S) . By the chain rule,

$$\frac{d}{dt} u(\varphi(t), \psi(t), \eta(t)) = u_x \frac{d\varphi}{dt} + u_y \frac{d\psi}{dt} + u_z \frac{d\eta}{dt}.$$

By Theorem 3.3, $\left(\frac{d\varphi}{dt}, \frac{d\psi}{dt}, \frac{d\eta}{dt}\right) = K(t)(P, Q, R)$ along γ . Hence

$$\frac{d}{dt} u(\varphi(t), \psi(t), \eta(t)) = K(t)(Pu_x + Qu_y + Ru_z)|_{(\varphi, \psi, \eta)}.$$

If u is a first integral, the left-hand side is identically 0 along every solution curve; since $K(t)$ is generically nonzero and solution curves pass through every point of the domain, $Pu_x + Qu_y + Ru_z = 0$ everywhere. Conversely, if this identity holds everywhere, the right-hand side vanishes along every solution curve, so u is constant along it, i.e. u is a first integral. ■ □

4.2 Functional Independence

Definition 4.3. Two C^1 functions u, v on an open set $G \subset \mathbb{R}^3$ are **functionally independent** if the only differentiable functions $H(u, v)$ such that $H(u(x, y, z), v(x, y, z))$ is constant on G are the constant functions H . Similarly, three functions u, v, w are functionally independent if the only $F(u, v, w)$ constant on G is a constant function.

Theorem 4.4. Two functions $u, v \in C^1(G)$ are functionally independent on G if and only if the matrix

$$\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \end{pmatrix}$$

has rank 2 throughout G . Similarly, three functions $u, v, w \in C^1(G)$ are functionally independent on G if and only if the Jacobian matrix

$$\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

has rank 3 throughout G .

Proof. We prove the two-function case; the three-function case is analogous.

($\Rightarrow$) Suppose u, v are functionally independent, and let $(a, b) \neq (0, 0)$ solve the linear system

$$(I): \quad a u_x + b v_x = 0, \quad a u_y + b v_y = 0, \quad a u_z + b v_z = 0.$$

Set $H(u, v) = au + bv$. Then $H(u(x, y, z), v(x, y, z))$ has identically vanishing partial derivatives on G (by the three equations of (I)), so it is constant on G . By functional independence, H

must be a constant function, which forces $a = b = 0$. Hence (I) has only the trivial solution, i.e. the matrix has rank 2.

($\Leftarrow$) Suppose the matrix has rank 2 and let $H(u, v)$ be such that $H(u(x, y, z), v(x, y, z))$ is constant on G . Differentiating with respect to x, y, z gives

$$H_u u_x + H_v v_x = 0, \quad H_u u_y + H_v v_y = 0, \quad H_u u_z + H_v v_z = 0.$$

Since the matrix $\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \end{pmatrix}$ has rank 2, this linear system in (H_u, H_v) has only the trivial solution $H_u = H_v = 0$. Hence H is constant, proving functional independence. ■ □

Theorem 4.5 (Structure of the set of first integrals). 1) Let u, v be two functionally independent first integrals of (S). Then every first integral w of (S) can be written as $w = F(u, v)$ for some C^1 function F .

2) Let u be a first integral of $\frac{dx}{P} = \frac{dy}{Q}$ (planar case). Then every first integral v of this system can be written as $v = H(u)$ for some C^1 function H .

Proof. We prove part 1). Let u, v, w be three first integrals of (S). By Theorem 4.2,

$$Pu_x + Qu_y + Ru_z = 0, \quad Pv_x + Qv_y + Rv_z = 0, \quad Pw_x + Qw_y + Rw_z = 0.$$

Since $(P, Q, R) \neq (0, 0, 0)$, the linear system

$$(II): \quad a u_x + b u_y + c u_z = 0, \quad a v_x + b v_y + c v_z = 0, \quad a w_x + b w_y + c w_z = 0$$

(in the unknowns a, b, c) admits the nontrivial solution $(a, b, c) = (P, Q, R)$. Hence its coefficient matrix is not of rank 3, i.e. the Jacobian $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ vanishes identically, so by Theorem 4.4, u, v, w are *not* functionally independent: there is a nontrivial relation

$$F(u(x, y, z), v(x, y, z), w(x, y, z)) = 0$$

for some non-identically-zero C^1 function F . Differentiating this relation with respect to x, y, z shows that (F_u, F_v, F_w) solves the transposed system analogous to (II); since u, v are independent, F_w cannot vanish without F_u, F_v also vanishing (which is impossible as F is nontrivial). By the implicit function theorem, since $F_w \neq 0$, we may locally solve for w as a function of u, v : $w = F(u, v)$. The proof of part 2) is entirely analogous, working with the 2×2 Jacobian. ■ □

4.3 Practical Methods for Finding First Integrals

Theorem 4.6 (Practical method). 1) To find a first integral u of $\frac{dx}{P(x, y)} = \frac{dy}{Q(x, y)}$: use the equality of the two ratios to find functions $A(x, y), B(x, y)$ such that

- (a) there exists u with $du = A dx + B dy$ (i.e. $A dx + B dy$ is an exact differential), and
- (b) $AP + BQ = 0$ (so that $A dx + B dy$ vanishes along solutions).

2) Similarly, to solve $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, seek two independent first integrals u, v by finding A, B, C such that $du = A dx + B dy + C dz$ is exact and $AP + BQ + CR = 0$.

Example 4.7. Solve $\frac{dx}{y} = \frac{dy}{x}$.

Solution. We have $\frac{dx}{y} = \frac{dy}{x} \Rightarrow x dx - y dy = 0$ (cross-multiplying), i.e. $d\left(\frac{x^2 - y^2}{2}\right) = 0$. Hence $u = x^2 - y^2$ is a first integral, and the solutions of (S) are the curves $x^2 - y^2 = a$, $a \in \mathbb{R}$. □

Remark 4.8 (Parametric method). One may also introduce an auxiliary parameter by setting $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = ds$, reducing (S) to the ODE system $\dot{x} = P$, $\dot{y} = Q$, $\dot{z} = R$, then integrate directly (as in the hyperbola example of Section 3).

5 First-Order Quasi-linear PDEs

5.1 General Form and Characteristic System

Definition 5.1. A quasi-linear first-order PDE for the unknown $z = z(x, y)$ is an equation of the form

$$f(x, y, z) \frac{\partial z}{\partial x} + g(x, y, z) \frac{\partial z}{\partial y} = h(x, y, z), \quad (E)$$

with $f, g, h \in C^1$. The **characteristic system** of (E) is

$$\frac{dx}{f(x, y, z)} = \frac{dy}{g(x, y, z)} = \frac{dz}{h(x, y, z)}. \quad (S_E)$$

5.2 Geometric Interpretation

Writing (E) as $(f, g, h) \cdot (z_x, z_y, -1) = 0$ shows that the vector field $V = (f, g, h)$ is tangent, at every point, to the graph surface $z = z(x, y)$ (since $(z_x, z_y, -1)$ is a normal vector to that surface).

Proposition 5.2. A C^1 surface $z = z(x, y)$ solves (E) if and only if it is a union of characteristic curves, i.e. of solution curves of (S_E) .

Proof. ($\Rightarrow$) Let z solve (E) and let $(x(t), y(t))$ solve $\dot{x} = f$, $\dot{y} = g$, with $\zeta(t) := z(x(t), y(t))$. By the chain rule,

$$\dot{\zeta} = z_x \dot{x} + z_y \dot{y} = z_x f + z_y g = h,$$

using (E). Hence $(x(t), y(t), \zeta(t))$ solves (S_E) , and lies on the surface: the surface is ruled by characteristics.

($\Leftarrow$) If the surface is ruled by characteristics, then at each point the tangent vector $V = (f, g, h)$ of the ruling curve lies in the tangent plane of the surface, whose normal is $(z_x, z_y, -1)$. Hence $V \cdot (z_x, z_y, -1) = 0$, i.e. $fz_x + gz_y - h = 0$, which is (E). ■ □

5.3 General Solution via First Integrals

Theorem 5.3 (Practical method for solving (E)). Assume h is not identically zero. To solve (E):

Step 1. Check whether $h(x, y, z) = 0$ itself defines a solution (an implicit solution obtained by solving $h = 0$ for z , if this is consistent).

Step 2. On the domain $h \neq 0$, find two independent first integrals u, v of the characteristic system (S_E) .

Step 3. Every solution of (E) is then given implicitly by

$$F(u(x, y, z), v(x, y, z)) = 0$$

for an arbitrary C^1 function F . (At least one of u, v must depend on z , otherwise the relation cannot be solved for z .)

Proof. By Proposition 5.2, a solution surface is a union of characteristic curves, i.e. solution curves of (S_E) . Since u, v are first integrals of (S_E) , they are constant along each such curve, so the surface lies in the level sets of the map (u, v) ; this is exactly the statement that $F(u, v) = 0$ for some F (using Theorem 4.5, since any first integral of (S_E) , in particular z restricted to the surface, is a function of u, v). Conversely, if $F(u, v) = 0$ defines $z = z(x, y)$ implicitly, differentiating with respect to x, y and using that u, v satisfy $fu_x + gu_y + hu_z = 0$ (Theorem 4.2) and similarly for v , one recovers $fz_x + gz_y = h$ exactly as in the proof of Theorem 3.5 of the quasilinear case. ■ □

Example 5.4. Solve $z z_x = z^2 - 1$.

Solution. Here $f = z$, $g = 0$, $h = z^2 - 1$.

- 1) $h = 0 \Leftrightarrow z = \pm 1$: these are two (trivial, constant) solutions.
- 2) For $z \neq \pm 1$, the characteristic system is $\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{z^2 - 1}$. From $dy = 0$: $y = C_1$, so $u = y$ is a first integral. From $\frac{dx}{z} = \frac{dz}{z^2 - 1}$: $dx = \frac{z dz}{z^2 - 1} = \frac{1}{2} d \ln |z^2 - 1|$, so $x - \frac{1}{2} \ln |z^2 - 1| = C_2$, giving the first integral $v = x - \frac{1}{2} \ln |z^2 - 1|$.
- 3) The general solution is $F(y, x - \frac{1}{2} \ln |z^2 - 1|) = 0$, or equivalently, solving for the second argument, $x - \frac{1}{2} \ln |z^2 - 1| = \varphi(y)$ for an arbitrary C^1 function φ , i.e.

$$z^2 = 1 + e^{2(x - \varphi(y))}.$$

□

5.4 The Linear Particular Case $f(x, y)z_x + g(x, y)z_y = 0$

Theorem 5.5. Let $f(x, y)z_x + g(x, y)z_y = 0$, with characteristic system $\frac{dx}{f(x, y)} = \frac{dy}{g(x, y)}$. If $u(x, y)$ is a first integral of this planar system, then every solution of the PDE is of the form $z = F(u)$ for an arbitrary C^1 function F .

Proof. This is the case $h \equiv 0$, f, g independent of z of Theorem 5.3: here z itself is a (trivial) first integral of the (now planar) characteristic system since $\dot{z} = h = 0$ along characteristics, and $u(x, y)$ is the other independent first integral, so by Theorem 4.5(2), $z = F(u)$. □

Example 5.6. Solve $y z_x - x z_y = 0$.

Solution. The characteristic system is $\frac{dx}{y} = \frac{dy}{-x} \Rightarrow x dx + y dy = 0 \Rightarrow x^2 + y^2 = C$, so $u = x^2 + y^2$ is a first integral. Hence every solution is of the form

$$z = F(x^2 + y^2)$$

for an arbitrary C^1 function F . □

6 The Cauchy Problem

6.1 Characteristic Curves and Non-Characteristic Data

Definition 6.1. The **characteristic curves** of (E) : $fz_x + gz_y = h$ are the solution curves of its characteristic system (S_E) .

Definition 6.2. A curve Γ is **non-characteristic** for (E) at a point if its tangent there is *not* parallel to the vector $V = f\vec{e}_1 + g\vec{e}_2 + h\vec{e}_3$. We say Γ is nowhere characteristic if this holds at every point of Γ .

Theorem 6.3 (Existence and uniqueness — Cauchy problem). *Through every curve Γ that is nowhere characteristic for (E) , there passes a unique solution of (E) . It is called the solution of the **Cauchy problem** relative to Γ .*

Sketch of proof. Parametrize Γ by $x = \varphi_0(s)$, $y = \psi_0(s)$, $z = \eta_0(s)$. Through each point of Γ , solve the characteristic system with these initial data; this generates a map $(s, t) \mapsto (x(s, t), y(s, t), z(s, t))$. Because Γ is nowhere characteristic, the tangent of Γ and the characteristic direction (f, g) are linearly independent at $t = 0$, i.e. the Jacobian $\frac{\partial(x, y)}{\partial(s, t)} \Big|_{t=0} \neq 0$.

By the inverse function theorem this map is locally invertible near Γ , so we may write $s = s(x, y)$, $t = t(x, y)$ and define $z(x, y) := z(s(x, y), t(x, y))$. As in Proposition 5.2, this z solves (E) and agrees with Γ 's data at $t = 0$. Uniqueness holds because the solution surface must be ruled by characteristics through Γ , and these are uniquely determined by the (locally Lipschitz) vector field (f, g, h) . ■

6.2 Practical Method: Elimination of the Parameter

Definition 6.4. Given two first integrals u, v of (S_E) and a parametrization $x = \varphi(t)$, $y = \psi(t)$, $z = \eta(t)$ of Γ , **eliminating** t between the equations $u(\varphi(t), \psi(t), \eta(t)) = a$ and $v(\varphi(t), \psi(t), \eta(t)) = b$ means finding the relation between a and b that must hold for every t .

Theorem 6.5 (Practical method for the Cauchy problem). *Let $(E) : fz_x + gz_y = h$ with characteristic system (S_E) , and let $\Gamma : x = \varphi(t)$, $y = \psi(t)$, $z = \eta(t)$ be a non-characteristic curve. Using two independent first integrals u, v of (S_E) , eliminate t between $u(\varphi(t), \psi(t), \eta(t)) = a$ and $v(\varphi(t), \psi(t), \eta(t)) = b$ to obtain a relation $H(a, b) = 0$. Then the solution of the Cauchy problem relative to Γ is*

$$H(u(x, y, z), v(x, y, z)) = 0.$$

(If $h \equiv 0$, recall that z itself may be taken as one of the two first integrals.)

Example 6.6. Find the solution of $E : yz_x - xz_y = 0$ passing through the ellipse $\Gamma : z = my$, $x^2 + y^2 = r^2$.

Solution. From the previous example, $u = z$ and $v = x^2 + y^2$ are first integrals ($u = z$ works since $\dot{z} = h = 0$ along characteristics). Parametrize Γ by $y = t$, $z = mt$, $x^2 = r^2 - t^2$. Then

$$u(\Gamma(t)) = mt = a \implies t = \frac{a}{m}, \quad v(\Gamma(t)) = r^2 = b \quad (\text{constant, independent of } t).$$

Eliminating t simply gives $b = r^2$ (no dependence on a remains), i.e. $H(a, b) = b - r^2$. Hence the Cauchy solution is

$$x^2 + y^2 - r^2 = 0,$$

i.e. z is undetermined by this relation alone — more precisely here since Γ lies exactly on the cylinder $x^2 + y^2 = r^2$ and $z = my$ prescribes $z = F(x^2 + y^2)$ only through this single value, the compatible solution is

$$z = m \frac{y}{\sqrt{x^2 + y^2}} r,$$

matching $z = my$ exactly when $x^2 + y^2 = r^2$. (In general, whenever Γ pins down F at a single value of $u = x^2 + y^2$ but varies z along it in a way not expressible as $F(x^2 + y^2)$, the Cauchy data must be compatible with the general solution $z = F(x^2 + y^2)$; here Γ is precisely one circle of such a cylinder, and $z = m \frac{yr}{\sqrt{x^2 + y^2}}$ is the natural choice reducing to $z = my$ on $x^2 + y^2 = r^2$.) $\square$

7 First-Order Nonlinear PDEs

7.1 Families of Surfaces and Envelopes

Definition 7.1. Let $\{S_\lambda\}$ be a one-parameter family of surfaces $F(x, y, z, \lambda) = 0$, with $F \in C^2$ and $F_{\lambda\lambda} \neq 0$. The **characteristic curve** of S_λ is the curve T_λ (if it exists) lying on S_λ defined by

$$T_\lambda : \quad F(x, y, z, \lambda) = 0, \quad \frac{\partial F(x, y, z, \lambda)}{\partial \lambda} = 0.$$

The surface generated by the curves T_λ as λ varies is called the **envelope** of the family $\{S_\lambda\}$, denoted Σ . The surfaces S_λ are tangent to Σ along T_λ . The equation of Σ is obtained by eliminating λ between the two equations defining T_λ .

Definition 7.2 (Two-parameter families). Let $\{S_{\lambda,\mu}\}$ be a family $F(x, y, z, \lambda, \mu) = 0$ with $F \in C^2$ and at least one second-order derivative in λ, μ nonzero.

- 1) A **characteristic point** of $S_{\lambda,\mu}$ is a point satisfying

$$F(x, y, z, \lambda, \mu) = 0, \quad \frac{\partial F}{\partial \lambda} = 0, \quad \frac{\partial F}{\partial \mu} = 0.$$

The set of characteristic points, when non-empty, forms a surface Σ called the **envelope** of the two-parameter family, obtained by eliminating λ, μ among these three equations. Every $S_{\lambda,\mu}$ is tangent to Σ at its characteristic points.

- 2) If, for a C^2 function φ , the one-parameter sub-family $F(x, y, z, \lambda, \varphi(\lambda)) = 0$ admits an envelope Σ_φ , this is called the **envelope with parameter** φ . The surfaces $S_{\lambda,\varphi(\lambda)}$ are tangent to Σ_φ .

7.2 The PDE Associated with a Two-Parameter Family

Write $z = \zeta(x, y)$, and set $p = z_x$, $q = z_y$.

Definition 7.3. The PDE associated with a two-parameter family $F(x, y, z, \lambda, \mu) = 0$ is the relation obtained by eliminating λ, μ from

$$F = 0, \quad p = -\frac{F_x}{F_z}, \quad q = -\frac{F_y}{F_z}$$

(these last two coming from implicit differentiation of $F = 0$ with respect to x and y). Every surface $S_{\lambda,\mu}$ of the family solves this PDE.

Definition 7.4 (Complete, general, and singular integrals). Let $E : G(x, y, z, p, q) = 0$ be a first-order PDE, and let $F(x, y, z, \lambda, \mu) = 0$ be a two-parameter family of solutions of E . This family is called a **complete integral** of E .

- Every envelope with one parameter φ (as above) is called a **general integral** of E .
- If the family admits a (two-parameter) envelope Σ , it is called a **singular integral** of E .

7.3 Constructing a Complete Integral: the Charpit–Lagrange Method

The envelope theory of Definition 7.4 tells us *what to do* with a complete integral once we have one — but not how to find it. This is furnished by **Charpit's method**.

Theorem 7.5 (Charpit–Lagrange equations). *Let $z \in C^2$ solve $G(x, y, z, p, q) = 0$ (with $p = z_x, q = z_y$), and let $(x(t), y(t))$ solve $\dot{x} = G_p, \dot{y} = G_q$. Writing $z(t), p(t), q(t)$ for the values along this curve, we have*

$$\dot{z} = pG_p + qG_q, \quad \dot{p} = -G_x - pG_z, \quad \dot{q} = -G_y - qG_z,$$

i.e., in symmetric form,

$$\frac{dx}{G_p} = \frac{dy}{G_q} = \frac{dz}{pG_p + qG_q} = \frac{dp}{-G_x - pG_z} = \frac{dq}{-G_y - qG_z}.$$

Proof. The identity for $\dot{z}$ is the chain rule: $\dot{z} = z_x \dot{x} + z_y \dot{y} = pG_p + qG_q$.

For $\dot{p}$: since $p = z_x(x(t), y(t))$,

$$\dot{p} = z_{xx} \dot{x} + z_{xy} \dot{y} = z_{xx} G_p + z_{xy} G_q.$$

Differentiating $G(x, y, z, z_x, z_y) = 0$ with respect to x (using Schwarz's theorem, $z_{yx} = z_{xy}$):

$$G_x + G_z z_x + G_p z_{xx} + G_q z_{xy} = 0 \implies z_{xx} G_p + z_{xy} G_q = -G_x - pG_z,$$

which, compared with the expression for $\dot{p}$, gives $\dot{p} = -G_x - pG_z$. The identity for $\dot{q}$ follows symmetrically by differentiating with respect to y . ■ □

Proposition 7.6. *Along solutions of the Charpit system, $G(x(t), y(t), z(t), p(t), q(t))$ is constant; in particular if $G = 0$ initially it remains 0.*

Proof.

$$\frac{dG}{dt} = G_x \dot{x} + G_y \dot{y} + G_z \dot{z} + G_p \dot{p} + G_q \dot{q},$$

and substituting the Charpit equations, every term cancels in pairs:

$$G_x G_p + G_y G_q + G_z (pG_p + qG_q) + G_p (-G_x - pG_z) + G_q (-G_y - qG_z) = 0. \quad \blacksquare$$

□

Theorem 7.7 (Practical procedure). *To build a complete integral of $G(x, y, z, p, q) = 0$:*

Step 1. *Write the Charpit–Lagrange system.*

Step 2. *Find an additional relation $H(x, y, z, p, q) = \lambda$, chosen as simply as possible (often directly from one of the five ratios, e.g. if the dp -equation gives $dp = 0$ then $p = \lambda$).*

Step 3. *Solve $\{G = 0, H = \lambda\}$ algebraically for p, q in terms of x, y, z, λ .*

Step 4. *Substitute into $dz = p dx + q dy$ and integrate to obtain the complete integral*

$$F(x, y, z, \lambda, \mu) = 0,$$

depending on two arbitrary constants λ, μ — exactly a two-parameter family of solutions in the sense of Definition 7.4.

Step 5. *If a Cauchy condition is prescribed, determine λ, μ directly; to obtain the general (resp. singular) integral, take the envelope with one parameter (resp. two parameters) of this complete integral, as in Section 7.1.*

7.4 Worked Examples

Example 7.8. Find a complete integral of $pq = z$.

Solution. Here $G = pq - z$, so $G_p = q$, $G_q = p$, $G_z = -1$, $G_x = G_y = 0$. The Charpit system gives

$$\frac{dp}{p} = \frac{dq}{q} \implies p = \lambda q$$

for a constant λ . Combined with $G = 0$: $\lambda q^2 = z \Rightarrow q = \sqrt{z/\lambda}$, $p = \sqrt{\lambda z}$. Then

$$dz = \sqrt{\lambda z} dx + \sqrt{z/\lambda} dy \implies \frac{dz}{\sqrt{z}} = \sqrt{\lambda} dx + \frac{dy}{\sqrt{\lambda}} \implies 2\sqrt{z} = \sqrt{\lambda} x + \frac{y}{\sqrt{\lambda}} + \mu.$$

The **complete integral** is

$$4z = \left(\sqrt{\lambda} x + \frac{y}{\sqrt{\lambda}} + \mu \right)^2, \quad \lambda, \mu \in \mathbb{R}, \lambda \neq 0.$$

□

Example 7.9 (Eikonal-type equation). Find a complete integral of $p^2 + q^2 = 1$, then its general and singular integrals.

Solution. With $G = p^2 + q^2 - 1$: $G_p = 2p$, $G_q = 2q$, $G_x = G_y = G_z = 0$, so $\dot{p} = \dot{q} = 0$, giving $p = \lambda$ constant, and $q = \pm\sqrt{1 - \lambda^2} =: \lambda'$. Then $dz = \lambda dx + \lambda' dy$, so the **complete integral** is the two-parameter family of planes

$$F(x, y, z, \lambda, \mu) : \quad z = \lambda x + \sqrt{1 - \lambda^2} y + \mu.$$

General integral: fix $\mu = \varphi(\lambda)$ for some C^1 function φ and eliminate λ between $F = 0$ and $F_\lambda = 0$; this produces a one-parameter envelope surface, tangent to each plane of the sub-family — a general (non-planar, developable) solution surface.

Singular integral: eliminate λ, μ between $F = 0$, $F_\lambda = 0$, $F_\mu = 0$. Here $F_\mu = 1 \neq 0$ always, so there is *no* characteristic point and hence *no* singular integral for this equation — consistent with the fact that $p^2 + q^2 = 1$ has no solution surface tangent to every plane of the complete integral simultaneously. □