Solutions d'Exercices (Serie 2)

Exercice 1:

① Soit. $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$x \longmapsto f(x) = \mathbb{1}_{[-1,1]}(x) = \begin{cases} 1 & \text{si } x \in [-1,1] \\ 0 & \text{sinon.} \end{cases}$$

$$\hat{f}(w) = \int_{-\infty}^{+\infty} f(x)\, e^{-iwx}\, dx$$

$$= \int_{-1}^{1} e^{-iwx}\, dx = \frac{+1}{-iw} \cdot e^{-iwx} \Big|_{-1}^{1}$$

$$= \frac{-1}{iw} \left[e^{-iw} - e^{iw} \right]$$

$$= \frac{-1}{iw} \left[\cos(-w) + i \sin(-w) - \cos(w) - i \sin(w) \right]$$

$$= \frac{-1}{iw} \left[\cancel{\cos(w)} - i \sin(w) - \cancel{\cos(w)} - i \sin(w) \right]$$

$$\hat{f}(w) = \frac{2 i \sin(w)}{iw} = 2 \cdot \frac{\sin(w)}{w}$$

② Soit $g : \mathbb{R} \longrightarrow \mathbb{R}$

$$t \longmapsto g(t) = \frac{\sin t}{t}$$

Nous avons: $\hat{f}(w) = \mathcal{F}(f(x)) \iff f(x) = \mathcal{F}^{-1}(\hat{f}(w))$

$$\ll 1 \gg$$

$$\iff f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(w)\, e^{iwx}\, dw$$

donc, comme

$$\mathcal{F}(f(x)) = \mathcal{F}\left(\mathbb{1}_{[-1,1]}(x) \right) = 2 \cdot \frac{\sin w}{w}$$

alors:

$$\mathbb{1}_{[-1,1]}(x) = f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2 \cdot \frac{\sin w}{w}\, e^{iwx}\, dw$$

$$\mathbb{1}_{[-1,1]}(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin w}{w}\, e^{iwx}\, dw \quad \cdots \quad (*)$$

Maintenant,

$$\hat{g}(w) = \int_{-\infty}^{+\infty} g(t)\, e^{-iwt}\, dt$$

$$= \int_{-\infty}^{+\infty} \frac{\sin t}{t}\, e^{-iwt}\, dt$$

$$\overset{(*)}{=} \pi \cdot \mathbb{1}_{[-1,1]}(-w)$$

$$= \begin{cases} \pi \cdot 1 & \text{si } -1 \leq -w \leq 1 \\ \pi \cdot 0 & \text{sinon} \end{cases}$$

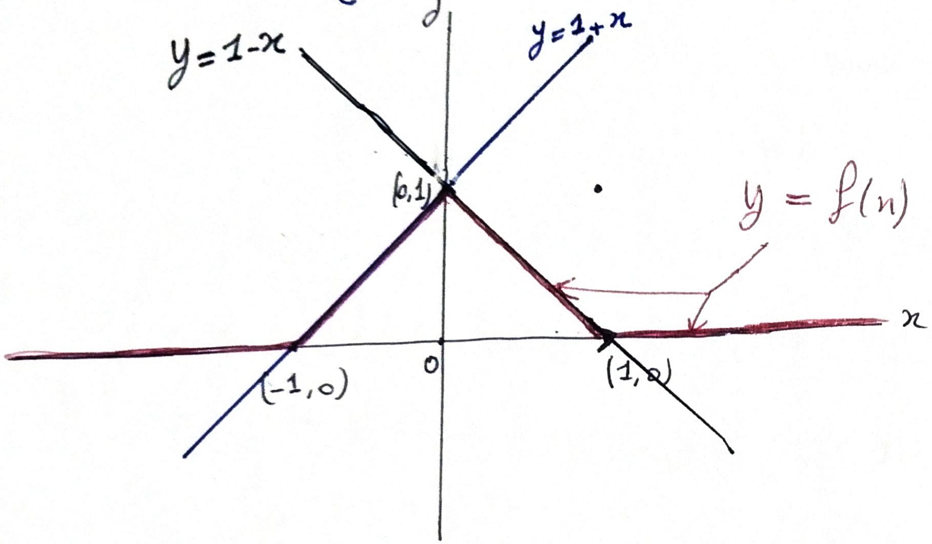
$$= \begin{cases} \pi & \text{si } -1 \leq w \leq 1 \\ 0 & \text{sinon} \end{cases} = \pi \cdot \mathbb{1}_{[-1,1]}(w)$$

$$\ll 2 \gg$$

Exercice 02:

① La représentation graphique de f:

On a $f(x) = \begin{cases} 1+x & \text{si } x \in [-1, 0] \\ 1-x & \text{si } x \in [0, 1] \\ 0 & \text{si } |x| > 1 \end{cases}$



② La transformée de Fourier de f:

$$\hat{f}(w) = \int_{-\infty}^{+\infty} f(x)\, e^{-iwx}\, dx$$

$$= \int_{-\infty}^{-1} 0 \cdot e^{-iwx} dx + \int_{-1}^{0} (1+x) e^{-iwx} dx + \int_{0}^{1} (1-x) e^{-iwx} + 0$$

$$= \underbrace{\int_{-1}^{0} (1+x) e^{-iwx} dx}_{I} + \underbrace{\int_{0}^{1} (1-x) e^{-iwx} dx}_{J}$$

$$I = \int_{-1}^{0} (1+x)\, e^{-iwx}\, dx$$

Posant $\begin{cases} u = 1+x \\ v' = e^{-iwx} \end{cases} \Rightarrow \begin{cases} u' = 1 \\ v = \frac{-1}{iw} e^{-iwx} \end{cases}$

$$I = \frac{-(1+x)}{iw} e^{-iwx} \Big|_{-1}^{0} - \int_{-1}^{0} \frac{-1}{iw} e^{-iwx} dx$$

$$= \frac{-1}{iw} \cdot 1 - 0 + \frac{1}{iw} \int_{-1}^{0} e^{-iwx} dx$$

$$= \frac{-1}{iw} + \frac{1}{iw} \cdot \left(\frac{-1}{iw}\right) \cdot e^{-iwx} \Big|_{-1}^{0}$$

$$= \frac{-1}{iw} + \frac{1}{w^2} \left(1 - e^{iw}\right)$$

$$I = \frac{iw + 1 - e^{iw}}{w^2}$$

$u = 1-x \rightarrow u' = -1$

$v' = e^{-iwx} \rightarrow v = \frac{-1}{iw} e^{-iwx}$

$$J = \int_{0}^{1} (1-x)\, e^{-iwx}\, dx$$

$$= \frac{-(1-x)}{iw} e^{-iwx} \Big|_{0}^{1} - \int_{0}^{1} (-1) \cdot \frac{-1}{iw} e^{-iwx} dx$$

$$= 0 - \left(\frac{-1}{iw} e^{0}\right) - \frac{1}{iw} \int_{0}^{1} e^{-iwx} dx$$

$$= \frac{1}{iw} - \frac{1}{iw}\left(\frac{-1}{iw} e^{-iwx}\Big|_0^1\right)$$

$$= \frac{1}{iw} - \frac{1}{w^2}\left(e^{-iw} - 1\right)$$

$$J = \frac{-iw - e^{-iw} + 1}{w^2}$$

Donc $\hat{f}(w) = I + J$

$$= \frac{2 - e^{iw} - e^{-iw}}{w^2}$$

$$= \frac{2 - 2\cos w}{w^2} = 2.\left(\frac{1-\cos w}{w^2}\right)$$

$$= 2.\left(\frac{1 - \cos 2.(\frac{w}{2})}{w^2}\right)$$

$$\boxed{\sin^2 t = \frac{1-\cos 2t}{2}}$$

$$= 2\left(\frac{2 \sin^2\frac{w}{2}}{w^2}\right)$$

$$\boxed{\hat{f}(w) = \frac{4 . \sin^2(\frac{w}{2})}{w^2}}$$

③ $\int_0^{\infty} \frac{\sin^4 x}{x^4}\, dx = ??$

Théorème de Parseval Plancherel (T.P.P.)

Si $f \in L^2$ alors: $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$

$f \in L^2 \iff \int_{-\infty}^{+\infty} f^2(x)\, dx < +\infty$

$$\int_{-\infty}^{+\infty} f^2(x)\, dx = \int_{-1}^{0} (1+x)^2 dx + \int_0^1 (1-x)^2 dx$$

$$= \frac{1}{3}(1+x)^3\Big|_{-1}^{0} - \frac{1}{3}(1-x)^3\Big|_0^1$$

$$= \frac{1}{3}(1-0) - \frac{1}{3}(0-1)$$

$$= \frac{2}{3} < +\infty \text{ donc } f \in L^2.$$

et $\|f\|_{L^2}^2 = \frac{2}{3}$

Alors d'après le Thé. P.P. $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$.

$$\|\hat{f}\|_{L^2}^2 = \int_{-\infty}^{+\infty} (\hat{f}(w))^2 dw$$

$$= \int_{-\infty}^{+\infty} \left(\frac{4 . \sin^2(\frac{w}{2})}{w^2}\right)^2 dw$$

$$= 16. \int_{-\infty}^{+\infty} \frac{\sin^4(\frac{w}{2})}{w^4} dw$$

$$\underset{t=\frac{w}{2}}{=} 16. \int_{-\infty}^{+\infty} \frac{\sin^4 t}{2^4 . t^4} 2.dt = 2\int_{-\infty}^{+\infty} \frac{\sin^4 t}{t^4}$$

$$\|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 \iff$$

$$\frac{2}{3} = 2\int_{-\infty}^{+\infty} \frac{\sin^4 t}{t^4}$$

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$$\Longrightarrow 2.2.\int_0^{\infty} \frac{\sin^4 t}{t^4} = \frac{2}{3}$$

d'où : $$\int_0^{\infty} \frac{\sin^4 t}{t^4} = \frac{1}{6}$$

fin

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