2.4 Exercises

Exercise 2.1

Are the following applications from E to F linear? If so, determine a basis of the kernel and a basis of the image.

1. $E = F = \mathbb{R}^2$; for all $(x, y) \in \mathbb{R}^2$:
$$f(x, y) = (2x + 3y, x)$$
2. $E = F = \mathbb{R}^2$; for all $(x, y) \in \mathbb{R}^2$:
$$f(x, y) = (y, x + y + 1)$$
3. $E = \mathbb{R}^3$, $F = \mathbb{R}$; for all $(x, y, z) \in \mathbb{R}^3$:
$$f(x, y, z) = x + 2y + z$$
4. $E = F = \mathbb{R}^2$; for all $(x, y) \in \mathbb{R}^2$:
$$f(x, y) = (x + y, xy)$$
5. $E = F = \mathbb{R}$; for all $x \in \mathbb{R}$:
$$f(x) = x^2$$

Exercise 2.2

Consider the linear map:
$$f : \mathbb{R}[X] \to \mathbb{R}[X]$$
$$P \mapsto P + XP',$$
where P' denotes the derivative of P.

1. Verify that f is indeed a linear transformation.
2. Determine the kernel $\ker f$ of this mapping.

3. Characterize the image $\text{Im}(f)$ of this transformation.

4. Is f injective? Surjective? Justify your answers.

Exercise 2.3

Let f be a function defined from $\mathbb{R}^3$ to $\mathbb{R}^3$ by:

$$f(x, y, z) = (2x + y,\ y - z,\ x - y).$$

1. Show that f is a linear map.

2. Find $\ker f$ and $\text{Im}\, f$, and determine their dimensions. Is f bijective?

3. Determine $f \circ f$.

Exercise 2.4

We denote $B = \{e_1, e_2, e_3\}$ the canonical basis of $\mathbb{R}^3$, and f the endomorphism in $\mathbb{R}^3$ defined by:

$$f(e_1) = -2e_1 + 2e_3, \quad f(e_2) = 3e_2, \quad f(e_3) = -4e_1 + 4e_3.$$

1. Let $u = (x, y, z) \in \mathbb{R}^3$. Calculate $f(u)$.

2. Determine a basis of $\ker f$.

3. Is f injective? Can it be surjective? Why?

4. Determine a basis of $\text{Im}\, f$. Deduce the rank of f.

5. Show that $\mathbb{R}^3 = \ker f \oplus \text{Im}\, f$.

Exercise 2.5

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map such that:

$$(1, 2, 0) \in \ker(f), \quad f(0, 0, 1) = (1, 0), \quad f(0, t, 0) = (t, t), \quad \forall t \in \mathbb{R}.$$

1. Determine the expression of $f(x, y, z)$.

2. Find a basis and the dimension of $\ker(f)$ and $\operatorname{Im}(f)$.

3. Find all vectors whose image is the vector $(0, 1)$. Is this set a vector subspace of $\mathbb{R}^3$?

Exercise 2.6

Let E, F, and G be three vector spaces over a field $\mathbb{K}$, and let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$ be linear maps. Prove that:

$$g \circ f = 0 \iff \operatorname{Im}(f) \subset \ker(g).$$

Exercise 2.7

Let E be a real vector space of dimension 3, and let $f \in \mathcal{L}(E)$ be an endomorphism such that:

$$f^2 \neq 0 \quad \text{and} \quad f^3 = 0.$$

Let $x_0 \in E$ such that $f^2(x_0) \neq 0$.

1. Show that the set $B = \{x_0, f(x_0), f^2(x_0)\}$ is a basis of E.

2. Let $g \in \mathcal{L}(E)$ such that $g \circ f = f \circ g$.

   (a) Show that there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that:

   $$g(x_0) = \alpha x_0 + \beta f(x_0) + \gamma f^2(x_0).$$

   (b) Show that:

   $$g = \alpha \mathrm{Id}_E + \beta f + \gamma f^2.$$

   (c) Deduce the set of endomorphisms of E that commute with f.