Solution of Tutorial work 03

Exercise 01:

- Determine wether S_1, S_2, S_3, S_4 are consistent or inconsistent.

1/
$$S_1 = \begin{cases} -w_1 + 2w_2 + w_3 = 2 & \text{--- (1)} \\ w_1 - 2w_3 = 1 & \text{--- (2)} \end{cases}$$

(2) gives $w_1 = 2w_3 + 1$ by replacing in (1) we get $w_2 = \frac{1}{2}(3 + w_3)$. So the solution depends on w_3 - it is a free variable

=) The system S_1 has infinitely many solutions

$$w_3 = t$$
$$w_2 = \frac{1}{2}(3 + t)$$
$$w_1 = 2t + 1$$

=) The system S_1 is consistent.

2/
$$S_2 = \begin{cases} -r_1 + 2r_2 + r_3 = 2 & \text{--- (1)} \\ r_2 - 2r_2 - r_3 = 1 & \text{--- (2)} \end{cases}$$

Let us add (1) and (2):

$$(1) + (2) \Rightarrow 0 = 3 \quad \text{- Contradiction}$$

we end up with $0 = 3$ which is impossible. This means the system has no solution (inconsistent)

3/

$$S_3 = \begin{cases} x_1 + 3x_2 - 2x_3 = 2 \\ \frac{1}{2}x_1 + 4x_2 + 8x_3 = 3 \\ 2x_1 - 2x_2 - x_3 = 1 \end{cases}$$

The system in augmented matrix is given by:

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ \frac{1}{2} & 4 & 8 & 3 \\ 2 & -2 & -1 & 1 \end{array}\right) \quad R_2 \to R_2 - \frac{R_1}{2} \quad \left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & \frac{5}{2} & 9 & 2 \\ 2 & -2 & -1 & 1 \end{array}\right)$$

$$R_3 \to R_3 - 2R_1 \left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & \frac{5}{2} & 9 & 2 \\ 0 & -8 & 3 & -3 \end{array}\right) \quad R_3 \to R_3 + \frac{16}{5}R_2 \left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & \frac{5}{2} & 9 & 2 \\ 0 & \cancel{8} & \frac{159}{5} & \frac{129}{5} \end{array}\right)$$

$$R_3 \to \frac{5}{159} R_3 \quad \left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & \frac{5}{2} & 9 & 2 \\ 0 & 0 & 1 & \frac{129}{159} = \frac{43}{53} \end{array}\right)$$

$$\Rightarrow \quad \begin{aligned} x_1 + 3x_2 - 2x_3 &= 2 \\ \frac{5}{2}x_2 + 9x_3 &= 2 \\ x_3 &= \frac{43}{53} \end{aligned}$$

$\Rightarrow$ the system S_3 is consistent.

4/

$$S_4 = \begin{cases} 3x + 2y + z = 2 \\ -x - 3y + 2z = 1 \\ 2x - y + 3z = 3 \end{cases}$$

The system it can be rewritten as follows:

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 2 \\ -1 & -3 & 2 & 1 \\ 2 & -1 & 3 & 3 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{ccc|c} -1 & -3 & 2 & 1 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 3 & 3 \end{array}\right)$$

$$\begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 2R_1 \end{array} \left(\begin{array}{ccc|c} -1 & -3 & 2 & 1 \\ 0 & -7 & 7 & 5 \\ 0 & -7 & 7 & 5 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2}$$

$$\longrightarrow \left(\begin{array}{ccc|c} -1 & -3 & 2 & 1 \\ 0 & -7 & 7 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{array}{r} -x - 3y + 2z = 1 \\ -7y + 7z = 5 \\ 0 = 0 \end{array}$$

The original system is equivalent to the last one

$\Rightarrow$ the system has many solutions (consistent)

Exercise 02 :

The linear system

$$\begin{aligned} b x_1 + 3(b-1)x_2 - 3x_3 &= 1 \\ x_1 + 2x_2 - 2b\,x_3 &= 5 \\ 3x_1 - 2x_2 - x_3 &= 2 \end{aligned}$$

it can be rewritten as $AX = b$:

$$\begin{pmatrix} b & 3(b-1) & -3 \\ 1 & 2 & -2b \\ 3 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

1- The system has a unique solution if $\det(A) \neq 0$

$$\det(A) = b \begin{vmatrix} 2 & -2b \\ -2 & -1 \end{vmatrix} - 3(b-1) \begin{vmatrix} 1 & -2b \\ 3 & -1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix}$$

$$\det(A) = -b(2+4b) - 3(b-1)(6b-1) + 3(2+6)$$

$$= -22b^2 + 19b + 21$$

$$\det(A) \Rightarrow -22b^2 + 19b + 21 = 0$$

$$\Delta = (19)^2 + 4(22)(21)$$

$$\Delta = 2209 \Rightarrow \sqrt{\Delta} = 47$$

$$b_1 = \frac{-19+47}{(-44)} = -\frac{7}{11}$$

$$b_2 = \frac{-19-47}{(-44)} = \frac{66}{44} = \frac{6}{4} = \frac{3}{2}$$

$\Rightarrow \det(A) \neq 0 \Rightarrow b \in \mathbb{R} - \left\{ -\frac{7}{11}, \frac{3}{2} \right\}$

2/ For $b = 1$ the system is written as:

$$\begin{pmatrix} 1 & 0 & -3 \\ 1 & 2 & -2 \\ 3 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 1 & 0 & -3 \\ 5 & 2 & -2 \\ 2 & -2 & -1 \end{vmatrix}}{\det(A)} \qquad x_2 = \frac{\begin{vmatrix} 1 & 1 & -3 \\ 1 & 5 & -2 \\ 3 & 2 & -1 \end{vmatrix}}{\det(A)} \qquad x_3 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & +2 \\ 3 & -2 & +5 \end{vmatrix}}{\det(A)}$$

From the previous we have:

$$\det(A) = -22b^2 + 19b + 21$$

$b = 1 \Rightarrow \det(A) = 18$

$$\begin{vmatrix} 1 & 0 & -3 \\ 5 & 2 & -2 \\ 2 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & -1 \end{vmatrix} - 3 \begin{vmatrix} 5 & 2 \\ 2 & -2 \end{vmatrix} = 36$$

$$\boxed{x_1 = \frac{36}{18} = 2}$$

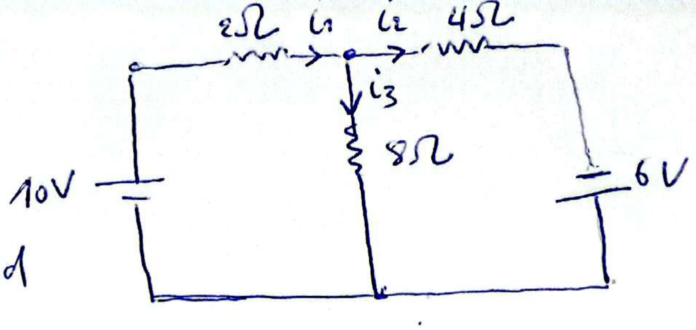
$$\begin{vmatrix} 1 & 1 & -3 \\ 1 & 5 & -2 \\ 3 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 5 & -2 \\ 2 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} = 33$$

$$\boxed{x_2 = \frac{33}{18} = \frac{11}{6}}$$

$$\begin{vmatrix} 1 & 0 & +1 \\ 1 & 2 & +2 \\ 3 & -2 & +5 \end{vmatrix} = \begin{vmatrix} 2 & +2 \\ -2 & +5 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 3 & +2 \end{vmatrix} = +8$$

$$\boxed{x_3 = \frac{+8}{18} = \frac{1}{3}}$$

Exercise 03 :



By using Kirchhoff's point and loop rules we get :

$$i_1 - i_2 - i_3 = 0$$
$$2i_1 + 8i_2 = 10$$
$$-8i_2 + 4i_3 = 6$$

In equivalent way

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & 8 & 0 & 10 \\ 0 & -8 & 4 & 6 \end{array}\right) R_2 \to R_2 - 2R_1$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 10 & 2 & 10 \\ 0 & -8 & 4 & 6 \end{array}\right) R_3 \to R_3 + \frac{8}{10} R_2 \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 10 & 2 & 10 \\ 0 & 0 & \frac{28}{5} & 14 \end{array}\right)$$

$$R_3 \longrightarrow \frac{5}{28} R_3 \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 10 & 2 & 10 \\ 0 & 0 & 1 & \frac{5}{2} \end{array}\right)$$

$$i_1 - i_2 - i_3 = 0 \quad — (1)$$

$$10 i_2 + 2 i_3 = 10 \quad — (2)$$

$$i_3 = \frac{5}{2} \quad — (3)$$

(3) in (2) $\Rightarrow$ $\boxed{i_2 = \frac{10-5}{10} = \frac{1}{2}\,A}$ — (4)

(3) and (4) in (1) $\Rightarrow$ $i_1 = \frac{1}{2} + \frac{5}{2} = 3\,A$

$$\Rightarrow \boxed{\begin{array}{l} i_1 = 3\,A \\ i_2 = \frac{1}{2}\,A \\ i_3 = \frac{5}{2}\,A \end{array}}$$