Solution of tutorial work 02   Algebra II

Exercise 01

1- $E_{ij} = i B_{ij} + j C_{ij}$

$E_{11} = 1 B_{11} + 1 C_{11} = 2$, $E_{12} = 2 + 12 = 13$ $E_{13} = 1 + 15 = 16$

$E_{21} = 2 \times 2 + 4 = 8$ $E_{22} = 2 \times 3 + 2 \times 3 = 12$ $E_{23} = 2 \times 4 + 3 \times 2$

$E_{31} = 3 \times 1 + 1 = 4$ $E_{32} = -2$ $E_{33} = 3 \times 2 + 3 = 9$

$$\Rightarrow \quad E = \begin{pmatrix} 2 & 13 & 16 \\ 8 & 12 & 14 \\ 4 & -2 & 9 \end{pmatrix}$$

$$- BA = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1+6+2 & 2+8+1 \\ 2+9+8 & 4+12+4 \\ 1-6+4 & 2-8+2 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 11 \\ 19 & 20 \\ -1 & -4 \end{pmatrix}$$

$$- A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \end{pmatrix}$$

$$- D_{ij} = B_{ji} + C_{ij} = B^T + C$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -2 \\ 1 & 4 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 6 & 5 \\ 4 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 8 & 6 \\ 6 & 6 & 0 \\ 2 & 6 & 3 \end{pmatrix}$$

$$(a) - (B+C)^2 \stackrel{?}{=} (B^2+C^2+2CB)$$

$$\text{①} \qquad\qquad \text{②}$$

$$\text{①} \Rightarrow \begin{pmatrix} 2 & 8 & 6 \\ 6 & 6 & 6 \\ 2 & 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 2 & 8 & 6 \\ 6 & 6 & 6 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 8 & 6 \\ 6 & 6 & 6 \\ 2 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4+48+12 & 16+48 & 12+48+18 \\ 12+36+12 & 48+36 & 36+36+18 \\ 4+6 & 16 & 12+9 \end{pmatrix}$$

$$= \begin{pmatrix} 64 & 64 & 78 \\ 60 & 84 & 90 \\ 10 & 16 & 21 \end{pmatrix}$$

$$\text{②} \Rightarrow B^2+C^2+2BC = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 6 & 5 \\ 4 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 6 & 5 \\ 4 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$+ 2\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 6 & 5 \\ 4 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4+1 & 2+6-2 & 1+6+1 \\ 2+6+4 & 4+9-8 & 2+12+4 \\ 1-4+1 & 2-6-2 & 1-8+1 \end{pmatrix} + \begin{pmatrix} 1+24+5 & 6+18+10 & 5+12+5 \\ 4+12+2 & 24+9+4 & 20+6+2 \\ 1+8+1 & 6+6+2 & 5+4+1 \end{pmatrix}$$

$$+ 2\begin{pmatrix} 1+8+1 & 6+6+2 & 5+4+1 \\ 2+12+4 & 12+9+8 & 10+6+4 \\ 1-8+1 & 6-6+2 & 5-4+1 \end{pmatrix} = \begin{pmatrix} 56 & 68 & 50 \\ 62 & 100 & 84 \\ -2 & 12 & 6 \end{pmatrix}$$

$$\text{②} \neq \text{①} \Rightarrow (B+C)^2 \neq B^2+C^2+2BC$$

(B) $\underset{1}{(2B)}A = B\underset{2}{(2A)}$?

$$1 \Rightarrow \left[2\begin{pmatrix}1 & 2 & 1\\ 2 & 3 & 4\\ 1 & -2 & 2\end{pmatrix}\right]\begin{pmatrix}1 & 2\\ 3 & 4\\ 2 & 1\end{pmatrix} = \begin{pmatrix}2 & 4 & 2\\ 4 & 6 & 8\\ 2 & -4 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\\ 2 & 1\end{pmatrix}$$

$$= \begin{pmatrix}2+12+2 & 4+16+2\\ 4+18+16 & 8+24+8\\ 2-12+8 & 4-16+4\end{pmatrix} = \begin{pmatrix}16 & 22\\ 38 & 40\\ -2 & -8\end{pmatrix}$$

$$2 \Rightarrow \begin{pmatrix}1 & 2 & 1\\ 2 & 3 & 4\\ 1 & -2 & 2\end{pmatrix}\begin{pmatrix}2 & 4\\ 6 & 8\\ 4 & 2\end{pmatrix} = \begin{pmatrix}2+12+4 & 4+16+2\\ 4+18+16 & 8+8+24\\ 2-12+8 & 4-16+4\end{pmatrix}$$

$$= \begin{pmatrix}16 & 22\\ 38 & 40\\ -2 & -8\end{pmatrix}$$

(C) $(BA)^T = A^T B^T =$

$$(BA)^T = \begin{pmatrix}9 & 11\\ 19 & 20\\ -1 & -4\end{pmatrix}^T = \begin{pmatrix}9 & 19 & -1\\ 11 & 20 & -4\end{pmatrix}$$

$$A^T B^T = \begin{pmatrix}1 & 3 & 2\\ 2 & 4 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ 2 & 3 & -2\\ 1 & 4 & 2\end{pmatrix} = \begin{pmatrix}1+6+2 & 2+9+8 & 1-6+4\\ 2+8+1 & 4+12+4 & 2-8+2\end{pmatrix}$$

$$= \begin{pmatrix}9 & 19 & -1\\ 11 & 20 & -4\end{pmatrix}$$

$\Rightarrow (BA)^T = A^T B^T$

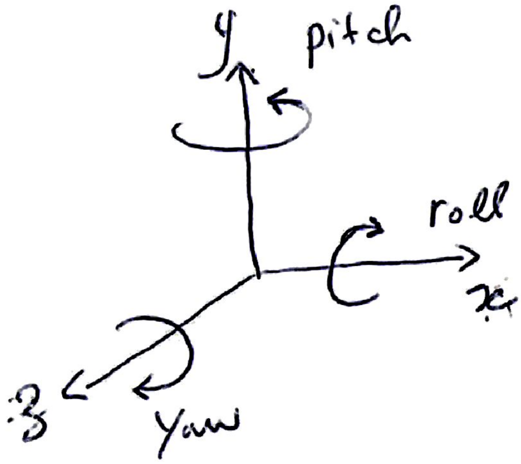
(d) $tr(B+C) = tr\begin{pmatrix} 2 & 8 & 6 \\ 6 & 6 & 6 \\ 2 & 0 & 3 \end{pmatrix} = 2+6+3 = 11$

$tr(B) + tr(C) = tr\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & -2 & 2 \end{pmatrix} + tr\begin{pmatrix} 1 & 6 & 5 \\ 4 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

$= (1+3+2) + (1+3+1) = 6+5 = 11$

$\Rightarrow tr(B+C) = tr(B) + tr(C)$

Exercise 02



pitch : rotation about y-axis

$\alpha = 30 \quad \sin\alpha = \frac{1}{2} \quad \cos\alpha = \frac{\sqrt{3}}{2}$

$$R_y = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

Roll : rotation about x-axis

$\beta = 45$

$\sin\beta = \cos\beta = \frac{\sqrt{2}}{2}$

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}$$

Yaw : rotation about z-axis

$\gamma = 60$

$\sin\gamma = \frac{\sqrt{3}}{2}$

$\cos\gamma = \frac{1}{2}$

$$R_z = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

* The coordinates of the point is given by :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_z R_x R_y \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= R_z R_x \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} \\ 1 \\ \frac{1}{2} - \frac{\sqrt{3}}{2} \end{pmatrix} = R_z \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} \\ \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) \\ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) \end{pmatrix}$$

$$= R_z \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} \\ \frac{\sqrt{2}}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \\ \frac{\sqrt{2}}{2}\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\left[1 + \sqrt{3} - \frac{\sqrt{6}}{2} - \frac{3\sqrt{2}}{2}\right] \\ \frac{1}{4}\left[\sqrt{3} + \frac{3}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}\right] \\ \frac{\sqrt{2}}{4}\left[3 - \sqrt{3}\right] \end{pmatrix}$$

Exercise 03 :

The matrix B is just a swap of R_2 and R_3 in A

and the matrix C is just a swap of two columns in B

then $\det(A) = -\det(B) = \det(C)$

$$\det(A) = 1 \times C_{11} + 2\, C_{12} + 5\, C_{13}$$

$$C_{11} = \begin{vmatrix} 4 & 7 \\ 1 & 2 \end{vmatrix} = 8 - 7 = 1 \qquad C_{12} = -\begin{vmatrix} 3 & 7 \\ 2 & 2 \end{vmatrix} = -(6 - 14) = 8$$

$$C_{13} = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = 3 - 8 = -5$$

* $\det(A) = 1 + 16 - 25 = -8$

* $5 \times A = \begin{pmatrix} 5 & 10 & 25 \\ 15 & 20 & 35 \\ 10 & 5 & 10 \end{pmatrix}$

* $\det(5A) = 5\,C_{11} + 10\,C_{12} + 25\,C_{13}$

$C_{11} = \begin{vmatrix} 20 & 35 \\ 5 & 10 \end{vmatrix} = 200 - 175 = 25$

$C_{13} = + \begin{vmatrix} 15 & 20 \\ 10 & 5 \end{vmatrix} = 75 - 200 = 125$

$C_{12} = - \begin{vmatrix} 15 & 35 \\ 10 & 10 \end{vmatrix} = 350 - 150 = 200$

$\det(5A) = 5 \times 25 + 10 \times 200 - 25 \times 125 = -1000$

$\det(5A) = 5^3 \times \det(A) = 125(-8) = 1000$

* $\det(B) = -\det(A) = 8$ and $\det(C) = -\det(B) = -8$

* $\det(A+B) = \begin{vmatrix} 2 & 4 & 10 \\ 5 & 5 & 9 \\ 5 & 5 & 9 \end{vmatrix}$

The addition of two matrices $A+B$ gives a matrix with two similar rows wich implies $\det(A+B) = 0$

$\Rightarrow \det(A+B) \neq \det(A) + \det(B)$

* <u>special case</u>: if B is a swap of two rows or columns in A the $\det(A+B) = \det(A) + \det(B)$

$$* \det(AB) = \left| \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 2 \\ 3 & 4 & 7 \end{pmatrix} \right|$$

$$= \begin{vmatrix} 1+4+15 & 2+2+20 & 5+4+35 \\ 3+8+21 & 6+4+28 & 15+8+49 \\ 2+2+6 & 4+1+8 & 10+2+14 \end{vmatrix}$$

$$= \begin{vmatrix} 20 & 24 & \\ 32 & 38 & 72 \\ 10 & 13 & 26 \end{vmatrix} = 20 \begin{vmatrix} 38 & 72 \\ 13 & 26 \end{vmatrix} - 24 \begin{vmatrix} 32 & 72 \\ 10 & 26 \end{vmatrix}$$

$$+ 44 \begin{vmatrix} 32 & 38 \\ 10 & 13 \end{vmatrix}$$

$$= 1040 - 24(112) + 44(36) = -64$$

$$= \det(A)\det(B) = (-8)(8) = -64$$

2/ $M = \begin{pmatrix} 1 & 2 & 5 \\ 9 & 7 & 13 \\ 2 & 1 & 2 \end{pmatrix}$

$$\det(M) = 1 \begin{vmatrix} 7 & 13 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 9 & 13 \\ 2 & 2 \end{vmatrix} + 5 \begin{vmatrix} 9 & 7 \\ 2 & 1 \end{vmatrix}$$

$$= 1 + 2(8) + 5(-5)$$

$$= -8$$

$$\det(M) = \det(A)$$

Exercise 04

$$-\ A = \begin{pmatrix} e & e+1 & e+2 \\ f & f+1 & f+2 \\ g & g+1 & g+2 \end{pmatrix} \xrightarrow{C_3 \to C_3 - C_2} \begin{pmatrix} e & e+1 & 1 \\ f & f+1 & 1 \\ g & g+1 & 1 \end{pmatrix}$$

$$\xrightarrow{C_2 \to C_2 - C_1} \begin{pmatrix} e & 1 & 1 \\ f & 1 & 1 \\ g & 1 & 1 \end{pmatrix}$$

- two rows are similar $\Rightarrow \det(A) = 0$.

- In the Matrix A $C_1 = 2C_2 - C_3 \Rightarrow$ the three columns are linearly dependent $\Rightarrow \det(A) = 0$

$$-\ B = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a+c & b+c & a+b \end{pmatrix} \xrightarrow{C_1 \to C_1 - C_2} \begin{pmatrix} 0 & 1 & 1 \\ a-b & b & c \\ a-b & b+c & a+b \end{pmatrix}$$

$$\xrightarrow{C_2 \to C_2 - C_3} \begin{pmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a-b & c-a & a+b \end{pmatrix}$$

$$\Rightarrow \det(B) = \begin{vmatrix} a-b & b-c \\ a-b & c-a \end{vmatrix} = (a-b)(c-a) + (b-a)(b-c)$$

$$= ac - a^2 - bc + \cancel{ba} + b^2 - bc - \cancel{ab} + ac$$

$$= b^2 - a^2 + 2ac - 2bc$$

Exercise 05

1- A is nonsingular if $\det(A) \neq 0$

$$\det(A) = 6\begin{vmatrix} 4 & 7 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 10 & 7 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 10 & 4 \\ 2 & 3 \end{vmatrix}$$

$$\det(A) = 6(8-21) - 2(20-14) + 5(30-8)$$
$$= -78 - 12 + 110 = 20$$

$\boxed{\det(A) = 20 \neq 0} \Rightarrow$ A is invertible and $\text{rank}(A) = 3$

2- Row reduction method to calculate $\det(A)$

$$A = \begin{pmatrix} 6 & 2 & 5 \\ 10 & 4 & 7 \\ 2 & 3 & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 5R_1} \begin{pmatrix} 6 & 2 & 5 \\ 0 & -11 & -3 \\ 2 & 3 & 2 \end{pmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{3}R_1} \begin{pmatrix} 6 & 2 & 5 \\ 0 & -11 & -3 \\ 0 & \frac{7}{3} & \frac{1}{3} \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + \frac{7}{33}R_2} \begin{pmatrix} 6 & 2 & 5 \\ 0 & -11 & -3 \\ 0 & 0 & -\frac{10}{33} \end{pmatrix}$$

$$\det(A) = 6(-11)\left(-\frac{10}{33}\right) = 20$$

3- $$\text{Adj}(A) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$\begin{pmatrix}6&2&5\\10&4&7\\2&3&2\end{pmatrix}\quad i=1,\ j=1 \qquad C_{11}=(-1)^{1+1}\begin{vmatrix}4&7\\3&2\end{vmatrix}=8-21=-13$$

$$\begin{pmatrix}6&2&5\\10&4&7\\2&3&2\end{pmatrix}\quad 1=i,\ j=2 \qquad C_{12}=(-1)^3\begin{vmatrix}10&7\\2&2\end{vmatrix}=-\begin{vmatrix}10&7\\2&2\end{vmatrix}=14-20=-6$$

$$\begin{pmatrix}6&2&5\\10&4&7\\2&3&2\end{pmatrix} \qquad C_{13}=(-1)^4\begin{vmatrix}10&4\\2&3\end{vmatrix}=30-8=22$$

$$\begin{pmatrix}6&2&5\\10&4&7\\2&3&2\end{pmatrix} \qquad C_{21}=-\begin{vmatrix}2&5\\3&2\end{vmatrix}=15-4=11$$

$$\begin{pmatrix}6&2&5\\10&4&7\\2&3&2\end{pmatrix} \qquad C_{22}=+\begin{vmatrix}6&5\\2&2\end{vmatrix}=12-10=2$$

$$\begin{pmatrix}6&2&5\\10&4&7\\2&3&2\end{pmatrix} \qquad C_{23}=-\begin{vmatrix}6&2\\2&3\end{vmatrix}=4-18=-14$$

$$\begin{pmatrix}6&2&5\\10&4&7\\2&3&2\end{pmatrix} \qquad C_{31}=\begin{vmatrix}2&5\\4&7\end{vmatrix}=14-20=-6$$

$$\begin{pmatrix}6&2&5\\10&4&7\\2&3&2\end{pmatrix} \qquad C_{32}=-\begin{vmatrix}6&5\\10&7\end{vmatrix}=50-42=8$$

$$\begin{pmatrix}6&2&5\\10&4&7\\2&3&2\end{pmatrix} \qquad C_{33}=+\begin{vmatrix}6&2\\10&4\end{vmatrix}=24-20=4$$

$$\operatorname{adj}(A) = \begin{pmatrix} -13 & -6 & 22 \\ 11 & 2 & -14 \\ -6 & 8 & 4 \end{pmatrix}^T = \begin{pmatrix} -13 & 11 & -6 \\ -6 & 2 & 8 \\ 22 & -14 & 4 \end{pmatrix}$$

$$A \cdot \operatorname{adj}(A) = \begin{pmatrix} 6 & 2 & 5 \\ 10 & 4 & 7 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} -13 & 11 & -6 \\ -6 & 2 & 8 \\ 22 & -14 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -78-12+110 & 66+4-70 & -36+16+20 \\ -130-24+154 & 110+8-98 & -60+32+28 \\ -26+18+44 & 22+6+28 & -12+24+8 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix} = \operatorname{diag}(\det(A), \det(A), \det(A))$$

$$= \det(A) \operatorname{diag}(1, 1, 1) = \det(A)\, I$$

$\frac{1}{\det(A)} A \operatorname{adj}(A) = I$ and we have $AA^{-1} = I$

then $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

$$= \frac{1}{20} \begin{pmatrix} -13 & 11 & -6 \\ -6 & 2 & 8 \\ 22 & -14 & 4 \end{pmatrix}$$

Exercise 06

1- the matrix representation of T

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3x+y+z\\2x+2y\\3z+4y\end{pmatrix} = x\begin{pmatrix}3\\2\\0\end{pmatrix} + y\begin{pmatrix}1\\2\\4\end{pmatrix} + z\begin{pmatrix}1\\0\\3\end{pmatrix}$$

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3 & 1 & 1\\2 & 2 & 0\\0 & 4 & 3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = A\begin{pmatrix}x\\y\\z\end{pmatrix}$$

the matrix representation of A is

$$A = \begin{pmatrix}3 & 1 & 1\\2 & 2 & 0\\0 & 4 & 3\end{pmatrix}$$

$$\det(A) = 3\begin{vmatrix}2 & 0\\4 & 3\end{vmatrix} - \begin{vmatrix}2 & 0\\0 & 3\end{vmatrix} + \begin{vmatrix}2 & 2\\0 & 4\end{vmatrix}$$

$$= 3(6) - 6 + 8 = 20$$

$\det(A) = 20 \neq 0 \Rightarrow \text{rank}(A) = 3$

2- $\det(A) \neq 0 \Rightarrow A$ is nonsingular

$\Rightarrow T$ is bijective

3- Determine T^{-1} and its [illegible]

$x' = 3x + y + 3$ —— (1)

$y' = 2x + 2y$ —— (2)

$z' = 3z + 4y$ —— (3)

$$2x' - 3y' = 2z - 4y \quad \text{— (4)}$$

$$(4) + (3) \Rightarrow 2x' - 3y' + z' = 5z$$

$$\boxed{z = \frac{2}{5}x' - \frac{3}{5}y' + \frac{1}{5}z'} \quad \text{— (5)}$$

$$\frac{3}{2} \times (4) \Rightarrow 3x' - \frac{9}{2}y' = 3z - 6y \quad \text{— (6)}$$

$$(6) - (3) = 3x' - \frac{9}{2}y' - z' = 3z - 6y - 3z - 4y$$

$$\boxed{y = -\frac{3}{10}x' + \frac{9}{20}y' - z'} \quad \text{— (7)}$$

by replacing (5) and (7) in (1) we get

$$x' = 3x - \frac{3x'}{10} + \frac{9}{20}y' + \frac{1}{10}z' + \frac{2}{5}x' - \frac{3}{5}y' + \frac{z'}{5}$$

$$x' = 3x + \frac{1}{10}x' - \frac{3}{20}y' + \frac{3}{10}z'$$

$$\frac{9}{10}x' + \frac{3}{20}y' - \frac{3}{10}z' = 3x$$

$$\boxed{x = \frac{3}{10}x' + \frac{1}{20}y' - \frac{1}{10}z'}$$

$$T^{-1}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{3}{10}x' + \frac{1}{20}y' - \frac{1}{10}z' \\ -\frac{3}{10}x' + \frac{9}{20}y' + \frac{1}{10}z' \\ \frac{2}{5}x' - \frac{3}{5}y' + \frac{1}{5}z' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{10} & \frac{1}{20} & -\frac{1}{10} \\ -\frac{3}{10} & \frac{9}{20} & \frac{1}{10} \\ \frac{2}{5} & -\frac{3}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

The matrix representation of T^{-1} is

$$B = \begin{pmatrix} \frac{3}{10} & \frac{1}{20} & -\frac{1}{10} \\ -\frac{3}{10} & \frac{9}{20} & \frac{1}{10} \\ \frac{2}{5} & -\frac{3}{5} & \frac{1}{5} \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 6 & 1 & -2 \\ -6 & 9 & 2 \\ 8 & -12 & 4 \end{pmatrix}$$

We can check that $B = A^{-1}$

$$AB = \frac{1}{20}\begin{pmatrix} 6 & 1 & -2 \\ -6 & 9 & 2 \\ 8 & -12 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \\ 2 & 2 & 0 \\ 0 & 4 & 3 \end{pmatrix}$$

$$= \frac{1}{20}\begin{pmatrix} 18+2 & 6+2-8 & 6-6 \\ -18+18 & -6+18+18 & -6+6 \\ 24-24 & 8-24+16 & 8+12 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix}$$

$$AB = BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Exercise 07

$\det(A) = 6 C_{11} + 3 C_{12} + 8 C_{13}$

$C_{11} = \begin{vmatrix} \frac{3}{2} & 4 \\ 5 & 1 \end{vmatrix} = \frac{3}{2} - 20 = -\frac{37}{2}$

$C_{12} = -\begin{vmatrix} 3 & 4 \\ 7 & 1 \end{vmatrix} = 28 - 3 = 25$

$C_{13} = \begin{vmatrix} 3 & 3/2 \\ 7 & 5 \end{vmatrix} = \frac{9}{2}$

All the 2×2 determinant are nonzero.

* $\det(A) = 0 \Rightarrow \operatorname{rank}(A) < 3$

Since at least one 2×2 determinant is nonzero, then $\operatorname{rank}(A) = 2$

Exercise 09

We have $B = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} = \{e_1, e_2, e_3\}$

$$B' = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\} = \{f_1, f_2, f_3\}$$

1- The matrix $S_{B' \to B}$ is determined as follows:

$$f_j = \sum_{i=1}^{3} s_{ij} \, e_i \quad \text{where } j = 1, 2, 3$$

$$f_1 = s_{11} e_1 + s_{21} e_2 + s_{31} e_3 \quad \text{— I}$$

$$f_2 = s_{12} e_1 + s_{22} e_2 + s_{32} e_3 \quad \text{— II}$$

$$f_3 = s_{13} e_1 + s_{23} e_2 + s_{33} e_3 \quad \text{— III}$$

$$\text{I} \Rightarrow \begin{pmatrix}1\\1\\0\end{pmatrix} = s_{11}\begin{pmatrix}1\\0\\0\end{pmatrix} + s_{21}\begin{pmatrix}0\\1\\0\end{pmatrix} + s_{31}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}s_{11}\\s_{21}\\s_{31}\end{pmatrix} \Rightarrow \begin{aligned} s_{11} &= 1 \\ s_{21} &= 1 \\ s_{31} &= 0 \end{aligned}$$

$$\text{II} \Rightarrow \begin{pmatrix}1\\0\\1\end{pmatrix} = s_{12}\begin{pmatrix}1\\0\\1\end{pmatrix} + s_{22}\begin{pmatrix}0\\1\\0\end{pmatrix} + s_{32}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\Rightarrow \begin{aligned} s_{12} &= 1 \\ s_{22} &= 0 \\ s_{32} &= 1 \end{aligned}$$

$$\text{III} \Rightarrow \begin{pmatrix}0\\1\\1\end{pmatrix} = s_{13}\begin{pmatrix}1\\0\\0\end{pmatrix} + s_{23}\begin{pmatrix}0\\1\\0\end{pmatrix} + s_{33}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{aligned} s_{13} &= 0 \\ s_{23} &= 1 \\ s_{33} &= 1 \end{aligned}$$

The matrix $S_{B' \to B}$:

$$S_{B' \to B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

2- Determine the coordinates of $[v]_{B'}$

we have $[v]_B = S_{B' \to B} [v]_{B'}$

$$\Rightarrow S^{-1}_{B' \to B} [v]_B = [v]_{B'}$$

$$S^{-1}_{B' \to B} = \frac{1}{\det(S_{B' \to B})} \operatorname{adj}(S_{B' \to B})$$

$$S^{-1}_{B' \to B} = \frac{1}{(-2)} \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow [v]_{B'} = S^{-1}_{B' \to B} [v]_B = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Second method:

$$\sum_{i=1}^{3} x_i e_i = \sum_{i=1}^{3} x'_i f_i \qquad [v]_{B'} = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}$$

$$1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = x'_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x'_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x'_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad x_1' + x_2' = 1 \quad \text{—— (1)}$$

$$x_1' + x_3' = 2 \quad \text{—— (2)}$$

$$x_2' + x_3' = 3 \quad \text{—— (3)}$$

$(1) - (2) \Rightarrow x_2' - x_3' = -1$ —— (4)

$(4) - (3) \Rightarrow 2x_2' = 2 \Rightarrow \boxed{x_2' = 1}$

From (1) we have $x_1' = 1 - x_2' = 1 - 1 = 0$

$$\Rightarrow \boxed{x_1' = 0}$$

From (2) we have $x_3' = 2 - x_1' = 2 - 0 = 2$

$$\boxed{x_3' = 2}$$

$$\Rightarrow \boxed{[v]_{\beta'} = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}$$