Tutorial: Comparison of Two or More Proportions (Applied Statistics)

Detailed Solutions

Exercise 1: Comparison Between Two Proportions

A study investigates a vaccine against a certain disease. Among the **vaccinated group** ($n_1 = 100$), **20 children** were infected. Among the **non-vaccinated group** ($n_2 = 120$), **42 children** were infected.

Questions

1) Compute the observed proportions p_1 and p_2.
2) State the hypotheses H_0 and H_1.
3) Compute the pooled proportion $\hat{p}$.
4) Compute the Z-statistic.
5) Conclude at $\alpha = 0.05$.

Solution

1) Observed proportions:

$$p_1 = \frac{20}{100} = 0.20, \quad p_2 = \frac{42}{120} = 0.35$$

2) Hypotheses:

$$H_0 : p_1 = p_2 \quad \text{(no difference)} \qquad H_1 : p_1 \neq p_2 \quad \text{(difference exists)}$$

3) Pooled proportion:

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{20 + 42}{220} = \frac{62}{220} = 0.2818$$

4) Test statistic:

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{p}(1 - \hat{p}) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}}$$

$$Z = \frac{0.20 - 0.35}{\sqrt{0.2818(1-0.2818)\left(\frac{1}{100}+\frac{1}{120}\right)}} = \frac{-0.15}{0.0609} = -2.46$$

5) Decision: At $\alpha = 0.05$, critical value $Z_{0.025} = 1.96$. Since $|Z| = 2.46 > 1.96$, we **reject** H_0.

Conclusion: There is a statistically significant difference between the two proportions. The vaccinated group shows a lower infection rate.

Exercise 2: Comparison of Three Proportions (Chi-Square Test)

An experiment compares three types of fertilizers (A, B, C) on wheat yield.

Fertilizer	Good Yield	Poor Yield	Total
A	18	12	30
B	15	15	30
C	21	9	30
Total	54	36	90

Questions

1) State the hypotheses.
2) Compute the expected frequencies E_{ij}.
3) Compute the Chi-square statistic.
4) Conclude at $\alpha = 0.05$.

Solution

1) Hypotheses:

$$H_0 : \text{Distribution of (Good/Poor) yield is the same for all fertilizers.}$$

$$H_1 : \text{At least one fertilizer differs.}$$

2) Expected frequencies:

$$E_{ij} = \frac{(\text{Row total}) \times (\text{Column total})}{N}$$

$$E_{\text{Good}} = \frac{30 \times 54}{90} = 18, \quad E_{\text{Poor}} = \frac{30 \times 36}{90} = 12$$

3) Chi-square statistic:

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

$$\chi^2 = \frac{(18-18)^2}{18} + \frac{(12-12)^2}{12} + \frac{(15-18)^2}{18} + \frac{(15-12)^2}{12} + \frac{(21-18)^2}{18} + \frac{(9-12)^2}{12}$$

$$\chi^2 = 0 + 0 + 0.5 + 0.75 + 0.5 + 0.75 = 2.50$$

4) Decision: Degrees of freedom $(r-1)(c-1) = (3-1)(2-1) = 2$. Critical value $\chi^2_{0.95,2} = 5.99$.

Since $2.50 < 5.99$, we **do not reject** H_0.

Conclusion: There is no significant difference among the fertilizers at $\alpha = 0.05$.

Exercise 3: Comparison of Three Success Rates (Chi-Square Test)

A study compares the success rate in mathematics in three schools:

School	Passed	Failed	Total
X	45	15	60
Y	40	20	60
Z	35	25	60
Total	120	60	180

Questions

1) State the hypotheses.

2) Compute the expected frequencies E_{ij}.

3) Compute the Chi-square statistic.

4) Conclude at $\alpha = 0.05$.

Solution

1) Hypotheses:

$$H_0 : \text{Success/failure distribution is identical in all schools.}$$

$$H_1 : \text{At least one school differs.}$$

2) Expected frequencies:

$$E_{\text{Passed}} = \frac{60 \times 120}{180} = 40, \quad E_{\text{Failed}} = \frac{60 \times 60}{180} = 20$$

3) Chi-square statistic:

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

$$\chi^2 = \frac{(45 - 40)^2}{40} + \frac{(15 - 20)^2}{20} + \frac{(40 - 40)^2}{40} + \frac{(20 - 20)^2}{20} + \frac{(35 - 40)^2}{40} + \frac{(25 - 20)^2}{20}$$

$$\chi^2 = 0.625 + 1.25 + 0 + 0 + 0.625 + 1.25 = 3.75$$

4) Decision: Degrees of freedom $(r - 1)(c - 1) = 2$. Critical value $\chi^2_{0.95,2} = 5.99$. Since $3.75 < 5.99$, we **do not reject** H_0.

Conclusion: There is no significant difference in success rates among the schools at $\alpha = 0.05$.

End of Exercises