Solution of the Assessment exam

Exercise 01

1) $\text{Span}\{u_1, u_2, u_3\} = \{\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \mid \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}$

$v \in \text{Span}\{u_1, u_2, u_3\} \Rightarrow v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$

$$\begin{pmatrix} 1 \\ 2 \\ a \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \lambda_1 + \lambda_2 = 1 & \text{— (1)} \\ \lambda_2 + \lambda_3 = 2 & \text{— (2)} \\ \lambda_2 + \lambda_3 = a & \text{— (3)} \end{cases}$$

by comparing (2) with (3) we get $\boxed{a = 2}$

2) If $a \neq 2$, $v \notin \text{span}\{u_1, u_2, u_3\}$

$\Rightarrow \{u_1, u_2, u_3\}$ does not span $\mathbb{R}^3$

Exercise 02

1) The vector $v = (1, b, 2) \in W \Rightarrow 3\underset{x}{(1)} - 2\underset{y}{(b)} - \underset{z}{(2)} = 0$

$\Rightarrow \boxed{b = \frac{1}{2}}$

2) $-2v \in W \Rightarrow \boxed{b = \frac{1}{2}}$

3) Since $\dim(W)$ = number of basis vector, then the basis must be determined to specify $\dim(W)$

- A set B is a basis of W if:

a/ B spans $W \Rightarrow W = \text{span}\{B\}$

b/ B is linearly independent

a/ $W = \{(x,y,z) \in \mathbb{R}^3 \mid 3x-2y-z=0\}$.

$W = \{(x,y,z) \in \mathbb{R}^3 \mid 3x-2y=z\}$

$W = \{(x,y,3x-2y) \mid x,y \in \mathbb{R}\}$

$W = \left\{ x\begin{pmatrix}1\\0\\3\end{pmatrix} + y\begin{pmatrix}0\\1\\-2\end{pmatrix} \mid x,y \in \mathbb{R}\right\}$

$\Rightarrow W = \text{Span}\left\{\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}$

b/ Is $\left\{\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}$ linearly independent?

$\alpha\begin{pmatrix}1\\0\\3\end{pmatrix} + \beta\begin{pmatrix}0\\1\\-2\end{pmatrix} = 0 \Rightarrow \begin{matrix}\alpha = 0\\ \beta = 0\\ 3\alpha+\beta=0\end{matrix}$

$\Rightarrow$ The set $\left\{\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}$ is linearly independent

$\Rightarrow \left\{\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}$ is a basis of W

$\Rightarrow \boxed{\dim(W) = 2}$

Exercise 03

1/ T is a linear transformation if:

a/ $T(v_1+v_2) = T(v_1) + T(v_2)$

b/ $T(\lambda v) = \lambda T(v)$

a- Let $v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$$T(v_1+v_2) = T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} x_1+x_2+2(y_1+y_2) \\ 8(x_1+x_2)-(z_1+z_2) \\ 4(x_1+x_2)+3(z_1+z_2) \end{pmatrix} = \begin{pmatrix} (x_1+2y_1)+(x_2+2y_2) \\ (8x_1-z_1)+(8x_2-z_2) \\ (4x_1+3z_1)+(4x_2+3z_2) \end{pmatrix}$$

$$= \begin{pmatrix} x_1+2y_1 \\ 8x_1-z_1 \\ 4x_1+3z_1 \end{pmatrix} + \begin{pmatrix} x_2+2y_2 \\ 8x_2-z_2 \\ 4x_1+3z_1 \end{pmatrix} = T(v_1) + T(v_2)$$

b/ $$T(\lambda v) = T\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} = \begin{pmatrix} (\lambda x)+2(\lambda y) \\ 8(\lambda x)-(\lambda z) \\ 4(\lambda x)+3(\lambda z) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda(x+2y) \\ \lambda(8x-z) \\ \lambda(4x+3z) \end{pmatrix} = \lambda\begin{pmatrix} x+2y \\ 8x-z \\ 4x+3z \end{pmatrix} = \lambda T(v)$$

$\Rightarrow$ T is a linear transformation.

2/ $\ast$ $Ker(T) = \{ v \in \mathbb{R}^3 \mid T(v) = 0 \}$

$$Ker(T) = \left\{ (x,y,z) \in \mathbb{R}^3 \mid \begin{pmatrix} x+2y \\ 8x-z \\ 4x+3z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$x + 2y = 0$ — (1)
$8x - z = 0$ — (2)
$4x + 3z = 0$ — (3)

(2) $-$ 2 (3) $\Rightarrow -z - 6z = 0 \Rightarrow z = 0$

By replacing in (2) $\Rightarrow x = 0$
" in (1) $\Rightarrow y = 0$

$$Ker(T) = \{ (0,0,0) \}$$

$\Rightarrow$ T is injective

- $Im(T) = \{ T(v) \mid v \in \mathbb{R}^3 \}$

$$= \{ (x+2y, 8x-z, 4x+3z) \mid (x,y,z) \in \mathbb{R}^3 \}$$

$$Im(T) = \left\{ x \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix} + y \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \mid x,y,z \in \mathbb{R} \right\}$$

$$Im(T) = Span \left\{ \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\}$$

and we have $\alpha \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = 0$, wich

gives
$\alpha + 2\beta = 0$ — (1)
$8\alpha - \gamma = 0$ — (2)
$4\alpha + 3\gamma = 0$ — (3)

$2(3)-(2) \Rightarrow \gamma = 0$

$\Rightarrow \alpha = 0$

$\Rightarrow \beta = 0$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\}$ is a basis of $Im(T)$

$\dim(Im(T)) = \dim(\mathbb{R}^3) \Rightarrow Im(T) = \mathbb{R}^3$

$\Rightarrow T$ is surjective

T is injective $+$ T is surjective $\Rightarrow T$ is bijective

3/ We have
$$x' = x + 2y \quad \text{--- } (4)$$
$$y' = 8x - z \quad \text{--- } (5)$$
$$z' = 4x + 3z \quad \text{--- } (6)$$

$(5) - 2(6) \Rightarrow y' - 2z' = -7z \Rightarrow \boxed{z = \frac{1}{7}(2z' - y')}$ $-(7)$

$3 \times (5) + (6) \Rightarrow 3y' - z' = 24x \Rightarrow \boxed{x = \frac{1}{24}(3y' - z')}$ $-(8)$

By replacing (7) and (8) in (4) we get

$$\boxed{y = \frac{1}{2}\left(x' + \frac{y'}{4} - \frac{z'}{12}\right)}$$

$$\Rightarrow \boxed{T^{-1}(x, y, z) = \left(\frac{1}{7}(2z - y), \frac{1}{24}(3y - z), \frac{1}{2}\left(x + \frac{y}{4} - \frac{z}{12}\right)\right)}$$