Solution of The second formative Exam!

Exercise 01 (7,25)

1/ $B^T = \begin{pmatrix} 1 & 4 \\ 3 & -2 \\ 2 & 3 \end{pmatrix}$ ✓ (0,5)

$$BA = \begin{pmatrix} 1 & 3 & 2 \\ 4 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -5 & 2 \\ 3 & -6 & -3 \\ -4 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 1+9-8 & -5-18+4 & 2-9+14 \\ 4-6-12 & -20+12+6 & 8+6+21 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -19 & 7 \\ -14 & -2 & 35 \end{pmatrix}$$ ✓ (01)

2/ det(A) using row reduction method

$$A = \begin{pmatrix} 1 & -5 & 2 \\ 3 & -6 & -3 \\ -4 & 2 & 7 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & -5 & 2 \\ 0 & 9 & -9 \\ -4 & 2 & 7 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 4R_1} \begin{pmatrix} 1 & -5 & 2 \\ 0 & 9 & -9 \\ 0 & -18 & 15 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{pmatrix} 1 & -5 & 2 \\ 0 & 9 & -9 \\ 0 & 0 & -3 \end{pmatrix}$$

$\det(A) = (1)(9)(-3) = -27$ ✓ (1,5) rank$(A) = 3$ ✓ (0,5)

3/ $\det(A) \neq 0 \Rightarrow A$ is invertible ✓ (0,25)

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A = -\frac{1}{27} \operatorname{adj} A$$

$$\operatorname{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$$C_{11} = \begin{vmatrix} -6 & -3 \\ 2 & 7 \end{vmatrix} = -42+6 = -36$$

$$C_{12} = -\begin{vmatrix} 3 & -3 \\ -4 & 7 \end{vmatrix} = -(21-12) = -9$$

$$C_{13} = \begin{vmatrix} 3 & -6 \\ -4 & 2 \end{vmatrix} = 6-24 = -18$$

$$C_{21} = -\begin{vmatrix} -5 & 2 \\ 2 & 7 \end{vmatrix} = 35+4 = 39$$

$$C_{22} = \begin{vmatrix} 1 & 2 \\ -4 & 7 \end{vmatrix} = 7+8 = 15$$

$$C_{23} = -\begin{vmatrix} 1 & -5 \\ -4 & 2 \end{vmatrix} = 20-2 = 18$$

$$C_{31} = \begin{vmatrix} -5 & 2 \\ -6 & -3 \end{vmatrix} = 15+12 = 27$$

$$C_{32} = -\begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix} = 3+6 = 9$$

$$C_{33} = \begin{vmatrix} 1 & -5 \\ 3 & -6 \end{vmatrix} = -6+15 = 9$$

$$\operatorname{adj}(A) = \begin{pmatrix} -36 & -9 & -18 \\ 39 & 15 & 18 \\ 27 & 18 & 9 \end{pmatrix}^T = \begin{pmatrix} -36 & 39 & 27 \\ -9 & 15 & 9 \\ -18 & 18 & 9 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{27}\begin{pmatrix} -36 & 39 & 27 \\ -9 & 15 & 9 \\ -18 & 18 & 9 \end{pmatrix}$$ ✓ 2,5

4/ $Bv = \begin{pmatrix} 1 & 3 & 2 \\ 4 & -2 & 3 \end{pmatrix} \begin{pmatrix} b \\ 2a \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\Rightarrow \quad \begin{aligned} b + 6a + 2 &= 1 \quad \text{— (1)} \\ 4b - 4a + 3 &= 3 \quad \text{— (2)} \end{aligned}$$

(2) gives $a = b$ by replacing in (1) we get

$$7b = -1 \Rightarrow \boxed{b = a = -\frac{1}{7}} \checkmark \text{ (01)}$$

Exercise 02 (03)

$T(x,y,z) = (x + 2ky,\ 2x - kz,\ 4x + 3z)$

$$T(x,y,z) = x \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + y \begin{pmatrix} 2k \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ -k \\ 3 \end{pmatrix}$$

The matrix representation A of T is

$$A = \begin{pmatrix} 1 & 2k & 0 \\ 2 & 0 & -k \\ 4 & 0 & 3 \end{pmatrix} \checkmark \text{ (1,5)}$$

T is not invertible if A is singular i.e. $\det(A) = 0$

$$\det(A) = -2k(6 + 4k) = 0$$

$$\begin{cases} k = 0 \\ \text{or} \\ 6 + 4k = 0 \Rightarrow k = -\frac{3}{2} \end{cases}$$

T is not invertible $\Rightarrow \boxed{k = 0, -\frac{3}{2}} \checkmark$ (1,5)