Chapter 2

Review of Probability Theory

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2.1 Random Experiment and Event

Definition 2.1.1. *(Random Experiment)*

A random experiment (r.e.) is any experiment whose outcome is governed by chance.

Definition 2.1.2. *(Sample Space)*

The set of all possible outcomes of a random experiment is called the sample space, denoted by Ω.

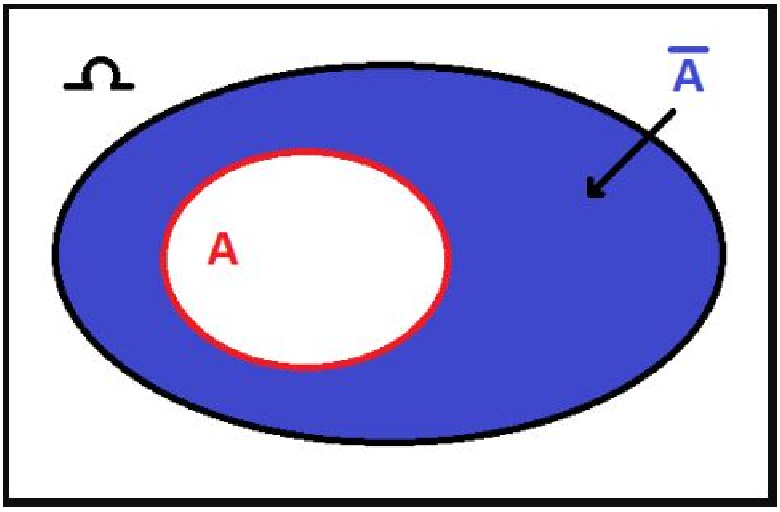
Definition 2.1.3. *(Event)*

An event in Ω is a subset of Ω.

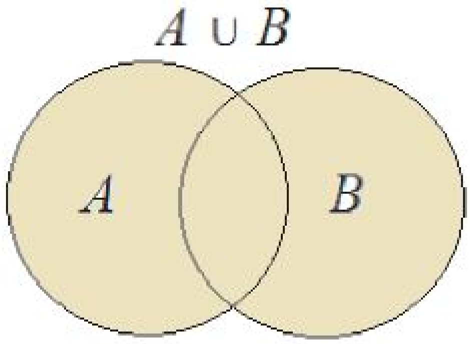
❶ An event is certain if it always occurs.

❷ An event is impossible if it never occurs.

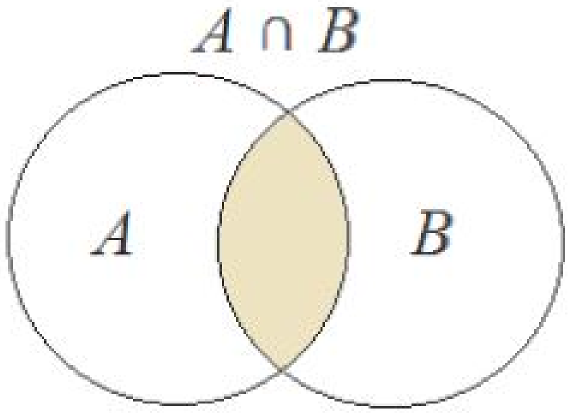
❸ The complement of event A is the event that occurs when A does not occur, denoted $\bar{A}$.



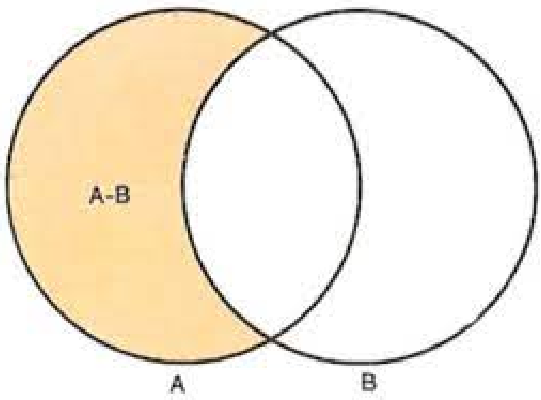
❹ The event $A \cup B$ occurs if either A or B occurs.



❺ The event $A \cap B$ occurs if both A and B occur.



❻ The event $A - B$ occurs if A occurs but not B.



❼ Events A and B are mutually exclusive (disjoint) if and only if $A \cap B = \emptyset$.

Example 2.1.1.

The random experiment of rolling a six-sided die numbered 1 to 6.

- *Sample space:*

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- *Event A: "getting the number 2"*

$$A = \{2\} \subset \Omega$$

- *Event B: "getting an even number"*

$$B = \{2, 4, 6\} \subset \Omega$$

- *Event: "getting an odd number": Complement event of B*

$$\bar{B} = \{1, 3, 5\}$$

- *Event: "getting a number less than 7": Certain event*

$$C = \{1, 2, 3, 4, 5, 6\}$$

- *Event: "getting a number greater than 8": Impossible event*
- *Event* $B - A = \{4, 6\}$
- *Event* $A \cup B = \{2, 4, 6\}$
- *Event* $A \cap B = \{2\}$
- *A and B are not disjoint because* $A \cap B \neq \emptyset$

Definition 2.1.4. *(Classical Definition of Probability)*

For each event A in a random experiment, the probability of A occurring is defined as:

$$P(A) = \frac{\text{number of cases where } A \text{ occurs}}{\text{total number of possible cases}} = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

Example 2.1.2.

The act of flipping a coin and observing the upper face is a random experiment.

- *The sample space is:*

$$\Omega = \{heads, tails\}$$

- *Event A: "getting heads"*

$$P(A) = \frac{1}{2}$$

- *Event B: "getting tails"*

$$P(B) = \frac{1}{2}$$

Definition 2.1.5. *(Probability)*

A probability is a function $P : \Omega \to [0, 1]$ such that, for every $A \in \Omega$:

❶ $P(\Omega) = 1$

❷ *For all mutually exclusive events A and B:*

$$P(A \cup B) = P(A) + P(B)$$

Properties 2.1.1. ❶ $P(\varnothing) = 0$

❷ $0 \leq P(A) \leq 1$

❸ $P(\bar{A}) = 1 - P(A)$

❹ $A \subset B \Rightarrow P(A) \leq P(B)$

❺ $P(A - B) = P(A) - P(A \cap B)$

❻ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

2.2 Conditional Probability

Definition 2.2.1.

For two events A and B such that $P(B) \neq 0$, the probability of A given B, denoted $P(A|B)$, is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example 2.2.1.

A class has 17 students:

- *8 study English,*
- *7 study German,*
- *2 study both.*

The probability that a student studies German given that they study English is:

$$P(L/A) = P_A(L) = \frac{P(L \cap A)}{P(A)}$$

we have

$$P(A) = \frac{8}{17}, \quad P(L) = \frac{7}{17}, \quad P(L \cap A) = \frac{2}{17}$$

then

$$P(L/A) = \frac{2/17}{8/17} = \frac{1}{4}$$

Remark 2.2.1.

- *If A and B are independent:*

$$P(A \cap B) = P(A)P(B)$$

Then,

$$P(A|B) = P(A), \quad P(B|A) = P(B)$$

2.3 Total Probability Formula

Definition 2.3.1.

Let E be a set. $B_1, B_2, ..., B_n$ *form a partition of E if :*

❶ $\forall\, i \in 1, ..., n;\ B_i \neq \emptyset$

❷ $\forall\, i \neq j;\ B_i \cap B_j = \emptyset$

❸ $B_1 \cup B_2 \cup ... \cup B_n = E$

Definition 2.3.2.

If the events $B_1, B_2, ..., B_n$ *form a partition of* Ω*, then:*

$$P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + ... + P(B_n)P(A/B_n)$$