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Mathematics 3 Course

Intended for second-year LMD students in Electromechanics and Mechanical Engineering

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Introduction

This document is a comprehensive and detailed resource covering the main concepts of *Mathematics III*, focusing on key topics in mathematical analysis and differential equations.
It begins with an in-depth exploration of single and multiple integrals, providing a complete understanding of integration methods, applications, and problem-solving techniques. Improper integrals are also discussed, focusing on cases with infinite limits or discontinuities.

A substantial section is devoted to differential equations, including first-order linear equations and their analytical solutions, along with their geometric interpretations. Second-order linear differential equations are studied in detail, with attention to particular solutions, boundary value problems and their practical applications.

The manuscript also includes a section on numerical and power series, examining their properties, convergence, and uses in various mathematical contexts. A thorough analysis of Fourier series follows, explaining their construction, convergence, and applications in engineering, physics, and applied mathematics.
Furthermore, the Fourier and Laplace transforms are presented in detail, highlighting their definitions, properties, and practical roles in solving differential equations and signal analysis problems.

Each chapter concludes with a collection of solved examples, designed to strengthen understanding and develop problem-solving skills through active learning.

Finally, I humbly ask Almighty God to bless this work, to make it sincere, beneficial, and enlightening for all readers — students, researchers, and anyone interested in this field. May its pages inspire clarity, wisdom, and the pursuit of knowledge for the advancement of science and humanity.

Contents

Chapter 1: Simple and Multiple Integrals

1.1. Review of the Riemann Integral and Calculation of Primitives

1.1.1. Subdivision

Let $[a,b]$ be a finite interval. We call a subdivision S of $[a,b]$ any finite ordered sequence $(t_i)_{0\leq i\leq n}$ within $[a,b]$.

$$a = t_0 \leq t_1 \leq \cdots \leq t_n = b$$

The *step size* of a subdivision S is defined as:

$$\rho(S) = max_{0\leq i\leq n-1}|t_{i+1} - t_i|$$

For a uniform subdivision of [a,b], the step size is $\frac{b-a}{n}$, and each point is calculated by the following equation:

$$t_i = a + i\frac{b-a}{n}, 0 \leq i \leq n$$

1.1.2. Riemann Integral

By definition, the Riemann integral of the function f is:

$$\int_a^b f(x)\,dx = Sup_S A^+(f,S) = Inf_S A^-(f,S)$$

A function $f: [a,b] \to \mathbb{R}$ is Riemann integral on $[a,b]$ if, for every $\varepsilon > 0$, there exists a subdivision S such that its Darboux sums satisfy

$$A^+(f,S) - A^-(f,S) \leq \varepsilon$$

The lower Darboux sum:

$$A^-(f,S) = \sum_{i=0}^{n-1} m_i(t_{i+1} - t_i)$$

With:

$$m_i = inf_{x\in[t_i,t_{i+1}[}\, f(x)$$

The upper Darboux sum:

$$A^+(f,S) = \sum_{i=0}^{n-1} M_i(t_{i+1} - t_i)$$

With:

$$M_i = sup_{x\in[t_i,t_{i+1}[}\, f(x)$$

If the subdivision is uniform, the Riemann sum takes the following form

$$R\big(f, S_{unif}\big) = \lim_{n\to+\infty} \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + i\frac{b-a}{n}) = \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + i\frac{b-a}{n})$$

1.1.3. Calculation of Primitives

Let f be a continuous and positive function on the interval $[a, b]$, the integral of this function is the area expressed in square units of the surface bounded by the curve C_f and the x-axis, between the points $x = a$ and $x = b$.

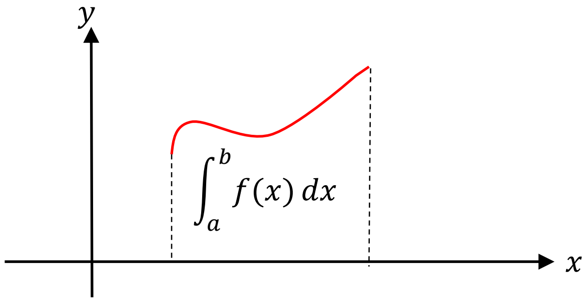

Fig.1: the curve of integral

We consider a function f continuous on an interval I. We say that a function F is a primitive of f on I if F is differentiable and its derivative is equal to:

$$F(x)' = f(x)$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

1.1.4. Primitives of Useful functions

We have c a real constant.

- $\int 1\,dx = x + c$
- $\int x^m\,dx = \frac{x^{m+1}}{m+1} + c$
- $\int \cos(x)\,dx = sin(x) + c$
- $\int \frac{1}{\sqrt{a^2-x^2}} dx = arcsin(\frac{x}{a}) + c$
- $\int \text{tang}(x)\,dx = -\ln(\cos(x)) + c$
- $\int \frac{1}{x} dx = \ln|x| + c$
- $\int e^x\,dx = e^x + c$
- $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} arctang(\frac{x}{a}) + c$
- $\int \frac{1}{sin^2(x)} dx = cotg(x) + c$
- $\int \frac{1}{cos^2(x)} dx = tang(x) + c$

The following functions all follow the rules of differentiation:

$$\int [u(x)]^n u'(x)\,dx = \frac{[u(x)]^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int e^{u(x)} u'(x)\,dx = e^{u(x)} + c$$

$$\int \frac{u'(x)}{u(x)} dx = \ln|u(x)| + c$$

$$\int \sin\big(u(x)\big)\,u'(x)\,dx = -\cos\big(u(x)\big) + c$$

$$\int \cos\big(u(x)\big)\, u'(x)\, dx = \sin\big(u(x)\big) + c$$

$$\int \frac{u'(x)}{\sqrt{u(x)}}\, dx = -\frac{1}{u(x)} + c$$

$$\int a^x\, dx = \frac{a^x}{\ln(a)} + c$$

$$\int u'(x) tang\big(u(x)\big)\, dx = -\ln(\cos(x)) + c$$

Some useful trigonometric formulas:

- $tang(x) = \frac{\sin(x)}{\cos(x)}$
- $cotg(x) = \frac{\cos(x)}{\sin(x)}$
- $cos^2(x) + sin^2(x) = 1$
- $cos^2(x) = \frac{1+\cos(2x)}{2}$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $sin^2(x) = \frac{1-\cos(2x)}{2}$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\cos(2x) = 2cos^2(x) - 1 = 1 - 2sin^2(x)$

1.1.5. Integration by parts:

Let u and v be two continuous functions on $[a, b]$, then:

$$\int_a^b u(x)v'(x)\, dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\, dx$$

Example 1.1: Compute the following integral:

$$\int_0^{\frac{\pi}{2}} x sin(x)\, dx$$

Answer: We set: $u(x) = x$, $u'(x) = 1$ $and\ v'(x) = \sin(x)$, $v(x) = -\cos(x)$

$$\int_0^{\frac{\pi}{2}} x sin(x)\, dx = [-x cos(x)]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} -\cos(x)\, dx = [-x cos(x)]_0^{\pi/2} + [\sin(x)]_0^{\pi/2} = 1$$

1.1.6. Variable change:

Let $f: [a, b] \to \mathbb{R}$ be a continuous function. Suppose there exists a function $v: [a, b] \to \mathbb{R}$ of class C^1 and a function u, continuous on $v([a, b])$, such that for all x in $[a, b]$, the following equality holds:

$$f(x) = u(v(x))v'(x)$$

Then:

$$\int_a^b f(x)\, dx = \int_{v(a)}^{v(b)} u(y)\, dy$$

In practical calculations, we set: $y = v(x), d'où\ dy = v'(x)dx$
Finally, we change the limits of the integral.

Example 1.2: calculate the following integral using a change of variable:

$$\int_0^2 \frac{2e^{\sqrt{x}}}{\sqrt{x}} dx$$

Answer: We use the following change of variable:

$$y = v(x) = \sqrt{x}$$

We obtain:

$$v(a) = 0, v(b) = \sqrt{2}\ et\ v'(x) = \frac{1}{2\sqrt{x}}$$

$$\int_0^2 \frac{2e^{\sqrt{x}}}{\sqrt{x}} dx = 4\int_0^2 \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx = 4\int_0^{\sqrt{2}} e^y\, dy = [e^y]_0^{\sqrt{2}} = 4(e^{\sqrt{2}} - 1)$$

1.1.7. Primitives of rational fractions:

A rational fraction is defined as the quotient of two polynomials. Most of the primitives that we can formally calculate can be done using simple changes of variables.

- Integral of the type: $\int \frac{1}{x+\alpha} dx, \alpha \in \mathbb{R}$

$$\int \frac{1}{x+\alpha} dx = \ln|x+\alpha| + c, c \in \mathbb{R}$$

- Integral of the type: $\int \frac{1}{(x+\alpha)^n} dx, \alpha \in \mathbb{R}\ and\ n \geq 1$

$$\int \frac{1}{(x+\alpha)^n} dx = \frac{1}{(1-n)(x+\alpha)^{n-1}} + c, c \in \mathbb{R}$$

1.2. Double Integral

Let $z = F(x, y)$ be a function defined in a closed and bounded domain D of the xoy plane. We divide the domain D arbitrarily into *n elementary subregions* of area $\Delta_{s_1}, \Delta_{s_2}, \dots, \Delta_{s_n}$. Now, we choose an arbitrary point $(P_k)_{1\le k\le n}$in each elementary domain.

Let $F(P_1), F(P_2), \dots, F(P_n)$ be the values of the function $F(x, y)$ at these points, and then form the products $F(P_k)\Delta_{s_k}$. The sum of these products is called a *Riemann sum* for the function $F(x, y)$ over the domain D, and it takes the following form:

$$\sum_{k=1}^{n} F(P_k)\Delta_{s_k} = F(P_1)\Delta_{s_1} + F(P_2)\Delta_{s_2} + \cdots + F(P_n)\Delta_{s_n}$$

The double integral of the function $F(x, y)$ over the domain D is defined as the limit of the integral sum when the largest of the domains $\Delta_{s_k} \to 0$. It is denoted as:

$$\iint F(x,y)\,dxdy = \iint F(P)\,ds = \lim_{max\Delta_{s_k}\to 0} \sum_{k=1}^{n} F(P_k)\,\Delta_{s_k}$$

1.2.1. Rule for calculating a double integral

Let $F(x,y)$ be a function defined and continuous in a closed domain D of $\mathbb{R}^2$, and this domain is such that:

$$\begin{cases} \text{si } a \le x \le b \quad \Rightarrow \quad f_1(x) \le y \le f_2(x) \\ \qquad\qquad ou \\ \text{si } c \le y \le d \quad \Rightarrow \quad g_1(y) \le x \le g_2(y) \end{cases}$$

Then, the double integral of the function F over D is calculated as follows:

$$\iint F(x,y)\,dxdy = \int_a^b [\int_{f_1(x)}^{f_2(x)} F(x,y)\,dy]dx = \int_c^d [\int_{g_1(y)}^{g_2(y)} F(x,y)dx]dy$$

Example 1.3: Calculate the following double integral: $I_1 = \iint \frac{x}{\sqrt{y}} dxdy$

Where the domain D is defined by: $D = \{(x,y) \in \mathbb{R}^2 / 0 < y \le x < 1\}$

Answer: Let $(x,y) \in D$, we choose $0 < x < 1$, and it follows that $0 < y \le x$. Thus,

$$I_1 = \int_0^1 \left[\int_0^x \frac{x}{\sqrt{y}} dy\right] dx = \int_0^1 x[2\sqrt{y}]_0^x\,dx = 2\int_0^1 (x\sqrt{x})\,dx = 2(\frac{2}{5}x^{5/2})_0^1 = \frac{4}{5}$$

1.2.2. Fubini's Theorem or Integration over a Rectangle

Let F(x, y) be a function defined and continuous in a rectangle $D = [a,b] \times [c,d]$ of $\mathbb{R}^2$:

Then,

$$\iint F(x,y)\,dxdy = \int_a^b [\int_c^d F(x,y)\,dy]dx = \int_c^d [\int_a^b F(x,y)dx]dy$$

Example 1.4: Compute the following double integral: $I_2 = \iint (x^2 + y)\,dxdy$

If the domain $D = [0,1] \times [1,2]$:

Answer: According to Fubini's Theorem, we have:

$$\iint (x^2+y)\,dxdy = \int_0^1 \left[\int_1^2 (x^2+y)\,dy\right] dx = \int_0^1 [x^2 y + \frac{1}{2}y^2]_1^2\,dx = \int_0^1 (x^2 + \frac{3}{2})\,dx$$

$$= \left(\frac{1}{3}x^3 + \frac{3}{2}x\right)_0^1 = \frac{11}{6}$$

1.2.3. Change of Variables in a Double Integral

A. Double Integral in Curvilinear Coordinates:

Suppose the integration variables x and y can be expressed as functions of new variables u and v as follows:

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

Where the functions $x(u,v)$ and $y(u,v)$ have continuous partial derivatives in a domain D' of the $uo'v$ plane, and the Jacobian of the transformation in the domain D' does not cancel:

$$J = \left|\frac{D(x,y)}{D(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

Then, the formula for transforming a double integral to curvilinear coordinates is as follows:

$$\iint F(x,y)\,dxdy = \iint F(x(u,v), y(u,v))|J|\,dudv$$

Example 1.5: Perform the change of variable indicated in the following integral:

$$I_4 = \int_0^1 dx \int_x^{2x} F(x,y)\,dy, \qquad \begin{cases} u = x+y \\ v = \dfrac{y}{x+y} \end{cases}$$

Answer: The new values of x and y in terms of (u,v) are:

$$\begin{cases} x = u - uv \\ y = uv \end{cases}$$

So,

$$J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

The new domain D' is:

$$D' = \{(u,v) \in \mathbb{R}^2, \frac{1}{2} \leq v \leq \frac{2}{3}, \qquad 0 \leq u \leq \frac{1}{1-v}\}$$

Then the integral I_4 is:

$$I_4 = \int_{\frac{1}{2}}^{\frac{2}{3}} dv \int_0^{\frac{1}{1-v}} uG(u,v)\,du$$

B. Double Integral in Polar Coordinates

When transitioning from rectangular (Cartesian) coordinates x and y to polar coordinates r and θ, we define:

$$\begin{cases} x = rcos\theta \\ y = rsin\theta \end{cases} \text{with } r > 0 \text{ and } 0 \leq \theta \leq 2\pi$$

We have:
$$J = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$$

The double integral of a function F in polar coordinates is:

$$\iint F(x,y)\,dxdy = \iint F(rcos\theta, rsin\theta)\,rdrd\theta$$

Example 1.6: Calculate $I_5 = \iint \frac{dxdy}{x^2+y^2}$ where the domain D is defined by :

$$D == \{(x,y) \in \mathbb{R}^2, 4 \leq x^2 + y^2 \leq 9\}$$

Answer: We have the polar coordinates as: $\begin{cases} x = rcos\theta \\ y = rsin\theta \end{cases}$

$$4 \leq x^2 + y^2 \leq 9 \,, 2 \leq r \leq 3, \text{hence } \; 0 \leq \theta \leq 2\pi$$

We have :

$$I_5 = \iint \frac{dxdy}{x^2 + y^2} = \iint \frac{rdrd\theta}{r^2cos^2\theta + r^2sin^2\theta} = \int_2^3 \frac{1}{r} dr \int_0^{2\pi} d\theta = 2\pi[\ln r]_2^3 = \pi \ln\frac{9}{4}$$

1.2.4. Area Calculation in $\mathbb{R}^2$

The area of a plane figure bounded by the domain D is calculated using the formula $A(D)$:

$$A(D) = \iint dxdy$$

Example 1.7: compute the area of A where domain D considered as follow:

$$D = \{(x, y) \in \mathbb{R}^2, 0 \leq x \leq 2, 1 \leq y \leq 5\}$$

Answer:

$$A(D) = \iint_D 1\, dxdy = \int_0^2 \left[\int_1^5 1\, dy\right] dx = \int_0^2 [y]_1^5\, dx = \int_0^2 4\, dx = [4x]_0^2 = 8$$

1.3. Triple Integral

Let $F\ (x, y, z)$ be a function defined in a closed and bounded domain D in the Ox, Oy, Oz. space. We arbitrarily divide the domain D into n elementary domains with volumes $\Delta V_1, \Delta V_2, \ldots, \Delta V_n$. Now, we choose an arbitrary point$(Pk)_{1\leq k\leq n}$ in each elementary domain. Let $F(P_1), F(P_2), \ldots, F(P_n)$ be the values of the function $F\ (x, y, z)$ at these points, and then form the products $F(P_k)\Delta V_k$.

The integral sum of the function $F(x, y, z)$ over the domain D is defined as a sum of the form

$$\sum_{k=1}^{n} F(P_k)\Delta V_k = F(P_1)\Delta V_1 + F(P_2)\Delta V_2 + \cdots + F(P_n)\Delta V_n$$

The triple integral of the function $F(x, y, z)$ over the domain D is defined as the limit of the integral sum when the largest of the volumes $\Delta V_k \to\ 0$. It is denoted by

$$\iiint F\ (x, y, z)dxdydz = \iiint F(P_k)dV = \lim_{max\Delta V_k \to 0} \sum_{k=1}^{n} F(P_k)\Delta V_k$$

1.3.1. Rule for Calculating a Triple Integral

Let F be a function defined and continuous in a closed domain D of $\mathbb{R}^3$, then:

$$\iiint F(x, y, z)\, dxdydz = \iint dxdy \int_{z_1}^{z_2} F(x, y, z)dz$$

The double integral over $dxdy$ is calculated on the domain T, where T is the orthogonal projection of D on to the xOy plane.

Example 1.8: calculate the triple integral $I_1 = \iiint (x + y + z)\, dxdydz$ the domain D is defined by:

$$D = \{(x, y, z) \in \mathbb{R}^3, x > 0, y > 0, z > 0, x + y + z \leq 2\}$$

Answer: The orthogonal projection of the domain D on to the xOy plane, where $z = 0$, is

$$T = \{(x, y) \in \mathbb{R}^2, x > 0, y > 0, x + y \leq 2\}$$

Additionally, $z_1 = 0\ et\ z_2 = 2 - x - y$ and $z_1 \leq z \leq z_2$. So,

$$I_1 = \iint dxdy \int_0^{2-x-y} (x + y + z)dz = \iint \left[(x + y)z + \frac{1}{2}z^2\right]_0^{2-x-y}$$

$$= \iint [2x + 2y - x^2 - y^2 - 2xy + \frac{1}{2}(2 - x - y)^2]dxdy$$

We have:

$$\begin{cases} 0 < y < 2 - x \\ 0 < x \leq 2 \end{cases}$$

Thus, the double integral of this function is:

$$I_1 = \int_0^2 \left[\int_0^{2-x} 2x + 2y - x^2 - y^2 - 2xy + \frac{1}{2}(2 - x - y)^2 dy\right] dx$$

$$= \int_0^2 [(2 - x)xy + y^2 - \frac{1}{6}y^3 - xy^2 + \frac{1}{2}(2 - x^2)y - \frac{1}{2}(2 - x)y^2]_0^{2-x}\, dx$$

$$= \int_0^2 [(2 - x)^2 - \frac{(2 - x)^3}{6}]\, dx$$

$$= [-\frac{(2 - x)^3}{3} + \frac{(2 - x)^4}{24}]_0^2 = \frac{1}{3} - \frac{1}{24} = \frac{7}{24}$$

1.3.2. Change of Variables in a Triple Integral

The change of variables in a triple integral allows for transitioning from the variables *x*, *y*, *z* to new variables *u*, *v*, *w* related to the original variables by the following relations:

$$x = x(u, v, w),$$

$$y = y(u, v, w),$$

$$z = z(u, v, w),$$

Where $x = x(u, v, w), y = y(u, v, w)\ and\ z = z(u, v, w)$, and their first partial derivatives are continuous functions in a domain D' . The Jacobian of the transformation in the domain D' is:

$$J = \left|\frac{D(x,y,z)}{D'(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$$

In this case, the formula for transforming a triple integral is

$$\iiint F(x,y,z)\,dxdydz = \iiint F(x(u,v,w), y(u,v,w), z(u,v,w))|J|dudvdw$$

1.3.3. Triple Integral in Spherical Coordinates

Let $M(x,y,z)$ be a point in space $\mathbb{R}^3$. We then have the following transformation formula:

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$

with $r > 0, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$.

the jacobien is:

$$J = \begin{vmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\theta \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\sin\varphi \end{vmatrix} = r^2\sin\varphi$$

It follows that: $dxdydz = r^2\sin\varphi\,drd\theta d\varphi$

Example 1.9: Calculate $I_2 = \iiint x^2\,dxdydz$, Where the domain D is defined by:

$$D = \{(x,y,z) \in \mathbb{R}^3, x^2 + y^2 + z^2 \leq a^2, a > 0\}$$

Answer: We perform a change of variables to spherical coordinates.

Now, let (x,y,z) be a point in D, we have:

$x^2 + y^2 + z^2 \leq a^2$, $r^2 \leq a^2$, and $0 \leq r \leq a$

then:

$$I_2 = \iiint x^2\,dxdydz = \int_0^a r^4\,dr \int_0^{2\pi} \cos^2\theta\,d\theta \int_0^\pi \sin^3\varphi\,d\varphi$$

$$= \left[\frac{1}{5}r^5\right]_0^a \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) \int_0^\pi \sin^3\varphi\,d\varphi$$

$$= \frac{4}{15}\pi a^5$$

1.3.4. Triple Integral in Cylindrical Coordinates

The transition from Cartesian coordinates to cylindrical coordinates is related by the following relations:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

with $r > 0 \; et \; 0 \le \theta \le 2\pi$. the value of jacobien is : $J = r$

$$\iiint F(x,y,z)\, dxdydz = \iiint F(r\cos\theta, r\sin\theta, z)\, r dr d\theta dz$$

Example 1.10: Compute the integral:

$$\iiint_V (x^2+y^2) dxdydz$$

Where the domain V is a cylinder defined by: $x^2+y^2 \le 4,\; 0 \le z \le 3$

Answer: we have:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$x^2+y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$\iiint_V (x^2+y^2)\, dxdydz = \int_{z=0}^{3}\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^2\, r dr d\theta dz$$

$$\iiint_V (x^2+y^2)\, dxdydz = \int_{z=0}^{3}\int_{\theta=0}^{2\pi}\left[\int_{r=0}^{2} r^3\, dr\right] d\theta dz = \int_{z=0}^{3}\int_{\theta=0}^{2\pi}\left[\frac{1}{4}r^4\right]_0^2 d\theta\, dz$$

$$= \int_{z=0}^{3}\left[\int_{\theta=0}^{2\pi} 4\, d\theta\right] dz = \int_{z=0}^{3} [4\theta]_{\theta=0}^{2\pi}\, dz = \int_{z=0}^{3} 8\pi\, dz = [8\pi]_0^3$$

$$= 24\pi$$

1.3.5. Volume Calculation

The volume of a body occupying the domain D is given by the following formula:

$$V = \iiint_D 1\, dxdydz$$

Example 1.11: let D a sphere $x^2+y^2+z^2 \le a^2$, we use the spherical coordinates:

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$

With, $r > 0, 0 \le \theta \le 2\pi, 0 \le \varphi \le \pi$, The Jacobien is: $dV = r^2\sin\varphi\, dr d\theta d\varphi$

Compute the volume of the sphere.

Answer: the volume of the sphere is:

$$V = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi}\int_{r=0}^{a} r^2\sin\varphi\, dr\, d\varphi d\theta = \int_{\theta=0}^{2\pi} 1\; d\theta \cdot \int_{\varphi=0}^{\pi} \sin\varphi\; d\varphi \cdot \int_{r=0}^{a} r^2 dr$$

$$= [\theta]_0^{2\pi} \cdot [-\cos\varphi]_0^{\pi} \cdot \left[\frac{1}{3}r^3\right]_0^a = \frac{4\pi a^3}{3}$$

Chapter 2: Improper Integrals

2.1. Definition of an Improper Integral

The integral $\int_a^b f(x)\,dx$ is called an improper integral if:

- $a = -\infty$ or $b = +\infty$ or both.
- We can find multiple points within the same interval as long as $a \leq x \leq b$. These points are called the singular points of $f(x)$.

2.2. Improper Integral of the First Kind

Let $f(x)$ be a bounded and integrable function over every finite interval $a \leq x \leq b$. Then, by definition:

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx$$

The integral $\int_a^{+\infty} f(x)\,dx$ is said to be convergent if $\lim_{b\to+\infty} \int_a^b f(x)\,dx$ exists.

The integral $\int_a^{+\infty} f(x)\,dx$ is said to be divergent if $\lim_{b\to+\infty} \int_a^b f(x)\,dx$ does not exist.

The same definition applies for:

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

Example 2.1: Compute the following improper integral:

$$\int_1^{+\infty} \frac{1}{x^2}\,dx$$

Answer:

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to+\infty} \left[-\frac{1}{t}+1\right] = 1$$

Where the limit is exist then, the improper integral is converge.

2.2.1. Improper Integral of the First Kind for Specific Functions

- The geometric or exponential integral: $\int_a^{+\infty} e^{-px}\,dx$, where t is a constant, is convergent if $p > 0$, and divergent if $p \leq 0$.

Example 2.2: Compute the following improper integral:

$$\int_1^{+\infty} e^{-2x}\,dx$$

Answer:

$$\int_{1}^{+\infty} e^{-2x}\,dx = \lim_{t\to+\infty}\int_{1}^{t} e^{-2x}\,dx = \lim_{t\to+\infty}\left[\frac{e^{-2x}}{-2}\right]_{1}^{t} = \lim_{t\to+\infty}\left[-\frac{e^{-2t}}{2}+\frac{e^{-2}}{2}\right] = \frac{e^{-2}}{2}$$

Since the limit exists, the improper integral converges ($p > 0$).

- The power integral of the first kind: $\int_{a}^{+\infty}\frac{dx}{x^p}$ where p is a constant and $a > 0$, is convergent if $p > 1$, and divergent if $p \le 1$. (This is known as the Riemann series.)

Example 2.3: Compute the following improper integral:

$$\int_{1}^{+\infty}\frac{1}{\sqrt{x}}dx$$

Answer:

$$\int_{1}^{+\infty}\frac{1}{\sqrt{x}}dx = \lim_{t\to+\infty}\int_{1}^{t}\frac{1}{\sqrt{x}}dx = \lim_{t\to+\infty}\left[2\sqrt{x}\right]_{1}^{t} = \lim_{t\to+\infty}\left[2\sqrt{t}-2\right] = +\infty$$

Since the limit diverges, the improper integral does not converge. ($p \le 1$).

- The power integral of the first kind: $\int_{0}^{1}\frac{dx}{x^p}$ where p is a constant and $a = 0$ is a singularity point, is convergent if $p < 1$, and divergent if $p \ge 1$. (This is known as the Riemann series.)

Example 2.4: Compute the following improper integral:

$$\int_{0}^{1}\frac{1}{x}dx$$

Answer:

$$\int_{0}^{1}\frac{1}{x}dx = \lim_{t\to0}\int_{t}^{1}\frac{1}{x}dx = \lim_{t\to0}[\ln x]_{t}^{1} = \lim_{t\to0}[-\ln t] = +\infty$$

If the limit is not exist then, the improper integral is diverge ($p \ge 1$).

2.2.2. Convergence Criteria for Improper Integrals of the First Kind

A. Comparison Criterion for Integrals with Non-Negative Integrands:

a) **Convergence:** Let $g(x) \ge 0$ for all $x \ge a$, and suppose that $\int_{a}^{+\infty} g(x)\,dx$ converges. Then, if $0 \le f(x) \le g(x)$ for all $x \ge a$, $\int_{a}^{+\infty} f(x)\,dx$ also converges.

Example 2.5: Compute the improper integral of $f(x)$:

$$\int_{1}^{+\infty}\frac{1}{x^2+x}dx$$

Answer: we propose $g(x) = \frac{1}{x^2}$

We compare between $f(x)$ and $g(x)$. For $x \ge 1$ we have $x^2 + x \ge x^2$ then the inverse is:

$\frac{1}{x^2+1} \leq \frac{1}{x^2}$, we conclude $0 \leq f(x) \leq g(x)$

The convergence of $\int_1^{+\infty} g(x)\,dx$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \left[-\frac{1}{x}\right]_1^t = 1$$

Then $\int_1^{+\infty} g(x)\,dx$ is converge.

According to the *comparison criterion*, if $g(x)$ converges, then $f(x)$ also converges.

b) **Divergence:** Let $g(x) \geq 0$ for all $x \geq a$, and suppose that $\int_a^{+\infty} g(x)\,dx$ diverges. Then, if $f(x) \geq g(x)$ for all $x \geq a$, $\int_a^{+\infty} f(x)\,dx$ also diverges.

Example 2.5: Compute the improper integral of (x) :

$$\int_1^{+\infty} \frac{2}{x}\,dx$$

We propose $g(x) = \frac{1}{x}$

We have: $f(x) = \frac{2}{x} \geq g(x) = \frac{1}{x}$

The divergence: $\int_1^{\infty} g(x)\,dx$:

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to+\infty} [\ln x]_1^t = \lim_{t\to+\infty} [\ln t] = +\infty$$

Then, $\int_1^{+\infty} g(x)\,dx$ is diverge.

According to the criteria of comparison, if $g(x)$ diverges then $f(x)$ also diverges.

B. Quotient Criterion for Integrals with Non-negative Integrands

a) If $f(x) \geq 0$ and $g(x) \geq 0$, and if $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = A \neq 0$ or ∞, then $\int_a^{+\infty} f(x)\,dx$ and $\int_a^{+\infty} g(x)\,dx$ either both converge or both diverge.

b) If $A = 0$ in (a) and if $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges.

c) If $A = \infty$ in (a) and if $\int_a^{+\infty} g(x)\,dx$ diverges, then $\int_a^{+\infty} f(x)\,dx$ diverges.

Example 2.6:

Compute the improper integral of $f(x)$:

$$\int_1^{+\infty} \frac{x+1}{x^3}$$

Answer: we propose $g(x) = \frac{1}{x^2}$

Compute the ratio:

$$\frac{f(x)}{g(x)} = \frac{x+1}{x^3} \cdot \frac{x^2}{1} = \frac{x+1}{x} = 1 + \frac{1}{x}$$

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} 1 + \frac{1}{x} = 1 \neq 0, \infty$$

$$\lim_{x\to+\infty} \int_1^{+\infty} \frac{1}{x^2}\, dx = 1$$

By the Quotient Criterion, since $g(x)$ converges, $f(x)$ also converges.

This criterion is related to the comparison criterion, of which it is a very useful alternative form. In particular, by taking $g(x) = \frac{1}{x^p}$, we obtain, based on the known behavior of this integral:

Theorem 1: let $\lim_{x\to+\infty} x^p f(x) = A$, then:

- $\int_a^{+\infty} f(x)\, dx$ converges if $p > 1$ and A est fini.
- $\int_a^{+\infty} f(x)\, dx$ diverges if $p \leq 1$ and $A \neq 0$ (A may be infinite).

Example 2.7: determine the convergence of this integral:

$$\int_1^{+\infty} \frac{1}{x^2}\, dx$$

Answer: we used this theorem: $\lim_{x\to+\infty} x^p f(x) = A$,

If we propose: $p = 2$, then:

$$\lim_{x\to+\infty} x^2 \cdot \frac{1}{x^2} = 1$$

According to the theorem, if $A = 1$ and $p > 1$ then, $f(x)$ is converge.

C. Absolute Convergence and semi-convergence:

$\int_a^{+\infty} f(x)\, dx$ is said to be absolutely convergent if $\int_a^{+\infty} |f(x)|\, dx$ converges. If $\int_a^{+\infty} f(x)\, dx$ converges but $\int_a^{+\infty} |f(x)|\, dx$ diverges, then $\int_a^{+\infty} f(x)\, dx$ is said to be semi-convergent.

2.3. Improper Integral of the Second Kind

If $f(x)$ is unbounded only at the endpoint $x = a$ of the interval $a \leq x \leq b$, then we define:

$$\int_a^b f(x)\, dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\, dx$$

We say that $\int_a^b f(x)\, dx$ is convergent if the limit exists.

We say that $\int_a^b f(x)\, dx$ is divergent if the limit does not exist.

If $f(x)$ is unbounded only at the endpoint $x = b$ of the interval $a \leq x \leq b$, then we define:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_a^{b-\varepsilon} f(x)\,dx$$

The integral $\int_a^b f(x)\,dx$ converges if the limit exists.

The integral $\int_a^b f(x)\,dx$ diverges if the limit does not exist.

If $f(x)$ is unbounded only at an interior point $x = x_0$ of the interval $a \le x \le b$. Then, we set:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon_1\to 0^+}\int_a^{x_0-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2\to 0^+}\int_{x_0+\varepsilon_2}^{b} f(x)\,dx$$

The integral $\int_a^b f(x)\,dx$ converges if the limit exists.

The integral $\int_a^b f(x)\,dx$ diverges if the limit does not exist.

Example 2.8: Compute the following improper integral:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

Answer:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0}\left[2\sqrt{x}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0}\left[2 - 2\sqrt{\varepsilon}\right] = 2$$

Since the limit is exist then, the improper integral is converge ($p < 1$).

2.3.1. Cauchy Principal Value

It may happen that the limits do not exist when ε_1 and ε_2 tend to zero independently. In this case, it is possible to set $\varepsilon_1 = \varepsilon_2 = \varepsilon$ so that:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_a^{x_0-\varepsilon} f(x)\,dx + \lim_{\varepsilon_2\to 0^+}\int_{x_0+\varepsilon}^{b} f(x)\,dx$$

If the limit exist then, the value limit called the Cauchy Principal Value.

Example 2.9: To define the principal value, we take symmetric limits around the singularity point $x_0 = 0$

$$\int_{-1}^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0^+}\left(\int_{-1}^{-\varepsilon}\frac{1}{x}\,dx + \int_\varepsilon^1 \frac{1}{x}\,dx\right) = \lim_{\varepsilon\to 0^+}(\ln\varepsilon - \ln\varepsilon) = 0$$

The Cauchy Principal Value exists and equals 0.

2.3.2. Improper Integral of the Second Kind for Specific Functions

- $\int_a^b \frac{dx}{(x-a)^p}\,dx$ converges if $p < 1$, and diverges if $p \ge 1$.
- $\int_a^b \frac{dx}{(b-x)^p}\,dx$ converges if $p < 1$, and diverges if $p \ge 1$.

Example 2.10: determine the convergence of this integral:

$$\int_0^1 \frac{dx}{\sqrt{1-x}}$$

Answer: we have $p = \frac{1}{2} < 1$ then, the integral is converge.

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = 2$$

Note: In the case where $p \leq 0$, the integrals are proper.

2.3.3. Convergence Criteria for Improper Integrals of the Second Kind

A. Comparison Criterion for Integrals with Non-negative Integrand

Convergence: Let $g(x) \geq 0$ for all $a < x \leq b$, and suppose that $\int_a^b g(x)\,dx$ converges.

Then, if $0 \leq f(x) \leq g(x)$ for all $a < x \leq b$, it follows that $\int_a^b f(x)\,dx$ also converges.

Example 2.11: determine the convergence of (x) :

$$\int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx$$

Answer: we propose $g(x) = \frac{1}{\sqrt{x}}$

We compare between $f(x)$ and $g(x)$:

$$1 + x \geq 1$$

$$(1+x)\sqrt{x} \geq \sqrt{x}$$

$$\frac{1}{(1+x)\sqrt{x}} \leq \frac{1}{\sqrt{x}}$$

Then,

$$0 \leq f(x) \leq g(x)$$

Verify the convergence of $g(x)$:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$$

So $g(x)$ is converge.

According to the comparison criteria, if $g(x)$ converges then, $f(x)$ converges.

Divergence: Let $g(x) \geq 0$ for all $a < x \leq b$, and suppose that $\int_a^b g(x)\,dx$ diverges. Then, if $f(x) \geq g(x)$ for all $a < x \leq b$, then $\int_a^b f(x)\,dx$ also diverges.

Example 2.12: determine the convergence of $f(x)$:

$$\int_0^1 \frac{3}{1-x}\,dx$$

We propose $g(x) = \frac{1}{1-x}$

We have, $f(x) \geq g(x)$

For any $\varepsilon > 0$,

$$\lim_{\varepsilon \to 1} \int_0^{1-\varepsilon} \frac{1}{1-x} dx = \lim_{\varepsilon \to 1} [\ln(1-x)]_0^{1-\varepsilon} = \lim_{\varepsilon \to 1} \ln(\varepsilon - 1) = \lim_{\varepsilon \to 0^+} \ln \varepsilon = +\infty$$

So, $\int_0^1 g(x)\, dx$ diverges.

According to the comparison criteria, if $g(x)$ diverges then, $f(x)$ diverges.

B. Quotient Criterion for Integrals with Non-Negative Integrand

a) If $f(x) \geq 0$ and $g(x) \geq 0$, for all $a < x \leq b$, if $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = A \neq 0$ or ∞, then $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$ both converge or both diverge.

b) If $A = 0$ dans (a) and if $\int_a^{+\infty} g(x)\, dx$ converges, then $\int_a^{+\infty} f(x)\, dx$ converges.

c) If $A = \infty$ dans (a) and if $\int_a^{+\infty} g(x)\, dx$ diverges, then $\int_a^{+\infty} f(x)\, dx$ diverges.

Example 2.13: Determine the convergence of $f(x)$:

$$\int_1^{+\infty} \frac{1}{\sqrt{x}} dx$$

Answer: we propose $g(x) = \frac{1}{x}$

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{x}{\sqrt{x}} = \lim_{x \to +\infty} \sqrt{x} = +\infty$$

Verify the convergence of $g(x)$:

$$\int_1^{+\infty} \frac{1}{x} dx = \lim_{t \to +\infty} \int_1^{+\infty} \frac{1}{x} dx = \lim_{t \to +\infty} [\ln x]_1^t = \lim_{t \to +\infty} [\ln t] = +\infty$$

Since $A = \infty$ and $\int g(x) dx$ diverges, we conclude that $\int f(x)\, dx$ also diverges.

This criterion is related to the comparison criterion, of which it is a very useful alternative form. In particular, by taking $g(x) = \frac{1}{(x-a)^p}$, we obtain, based on the known behavior of this integral:

Theorem 2 : let $\lim_{x \to +\infty} (x-a)^p f(x) = A$, then :

- $\int_a^b f(x)\, dx$ converges if $p < 1$ and if A is finite.
- $\int_a^b f(x)\, dx$ diverges if $p \geq 1$ and if $A \neq 0$ (A is infinite).

Example 2.14: determine the convergence of this integral:

$$\int_1^{+\infty} \frac{1}{(x-1)^{1/2}} dx$$

Answer: we used this theorem: $\lim_{x \to +\infty} (x-a)^p f(x) = A$, If we propose: $p = \frac{1}{2}, a = 1$then:

$$\lim_{x\to+\infty} (x-1)^{1/2} \cdot \frac{1}{(x-1)^{1/2}} = 1$$

According to the theorem, if $A = 1$ and $p < 1$ then, $f(x)$ is converge.

Theorem 3 : let $\lim_{x\to+\infty} (b-x)^p f(x) = B$, then:

- $\int_a^b f(x)\,dx$ converges if $p < 1$ and if B is finite.
- $\int_a^b f(x)\,dx$ diverges if $p \geq 1$ and if $B \neq 0$ (A is infinite).

C. Absolute and Conditional Convergence

$\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f(x)|\,dx$ converges. If $\int_a^b f(x)\,dx$ converges but $\int\int_a^b |f(x)|\,dx$ diverges, then $\int\int_a^b f(x)\,dx$ is said to be conditionally convergent.

Chapter 3: Differential Equations

3.1. Generalities:

3.1.1. Definition:

We call a differential equation (D.E.) of order n any equation of the form:

$$F\left(x, y(x), y'(x), \dots, y^n(x)\right) = 0$$

Where F is a function of $(n + 2)$ variables.

$y : I \rightarrow \mathbb{R}$ is a function that is n times differentiable on I.

Example 3.1: $F\left(x, y(x), y'(x)\right) = 0$ Therefore,

$$y'(x) + y(x) = 0$$

$$y(x) = y$$

$$y' + y = 0$$

3.2. Linear Differential Equation:

A differential equation of order n is said to be linear if it is of the form:

$$(E): \quad a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^n = g(x)$$

$g(x)$: Second Member

$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^n$: First Member.

Where a_n and $g(x)$ are continuous functions on $I \subset \mathbb{R}$.

Example 3.2: An example of a first-order linear differential equation (because it contains only one derivative) is:

$$a_0(x)y + a_1(x)y' = g(x)$$

if $(x) = 0$, $\forall\, x \in I$ Therefore : Equation (1) is called a homogeneous linear differential equation, and it is denoted by (E_0)

if $\forall\, i \in \{0,1, \dots, n\}$ where $a_i(x)$: is constante (1), $\forall\, x \in I$ Therefore (1) is called n-order linear differential equation with constant coefficients.

3.2.1. Proposition :

If y_1 and y_2 are two solutions of the homogeneous linear differential equation (E_0) of order n , then, $\forall\, \alpha, \beta \in \mathbb{R}$, we have $\alpha y_1 + \beta y_2$ is also a solution of this equation.

Proof:

$$\begin{cases} \alpha \times [a_0(x)y_1 + a_1(x)y_1' + \cdots + a_n(x){y_1}^n = 0] \\ \beta \times [a_0(x)y_2 + a_1(x)y_2' + \cdots + a_n(x){y_2}^n = 0] \end{cases}$$

Therefore $\forall\, \alpha, \beta \in \mathbb{R}$ we have :

$a_0(x)(\alpha y_1 + \beta y_2) + \cdots + a_n(x)(\alpha {y_1}^n + \beta {y_2}^n) = 0$

3.3. First-Order Linear Differential Equation

3.3.1. Definition

A first-order linear differential equation is of the form:

$$f(x)y' + g(x)y = h(x)$$

Where f, g, and h are continuous functions on an interval $I \subset \mathbb{R}$.

3.3.2. Theorem

Let the equation (E) $y' + a(x)y = b(x)$, be a first-order linear differential equation (F.O.L.D.E).

The equation $y' + a(x)y = 0$ is the corresponding homogeneous differential equation (H.D.E).

Let $A: I \to \mathbb{R}$ be the primitive of $a(x)$ (that is, $(A'(x) = a(x)$).

Then the solutions of the equation (E) on I are the functions $y: I \to \mathbb{R}$ defined by:

$y(x) = y_0(x) + y_p(x) = ke^{-A(x)} + e^{-A(x)}\left[\int b(x)e^{A(x)}\,dx\right] \quad , k \in \mathbb{R}$

$y_0(x)$: The solution of the homogeneous differential equation (H.D.E.)

$y_p(x)$: The particular solution.

Example 3.3: Solve the differential equation:

$$\text{(E):} \quad 5y' + 20y = 0$$

Answer: the D.E is simplified as follow: $y' + 4y = 0$

The primitive is: $a(x) = 4$, then , $A(x) = \int 4\,dx = 4x$

Therefore, the solutions of (E)

$$y(x) = ke^{-4x} \quad ,where\ \ k \in \mathbb{R}$$

Example 3.4: Solve the differential equation:

$$y' = 3x^2y + 2x^2$$

Answer: the D.E is simplified as follow: $y' - 3x^2y = 2x^2$

Let us find the solutions of (H.D.E)

$$(E_0): \qquad y' - 3x^2y = 0$$

$a(x) = -3x^2$ then : $A(x) = \int -3x^2\,dx = -x^3$

Then, $y_0 = ke^{x^3}\ where, k \in \mathbb{R}$

Let us find y_p:

$$y_p = \left[\int b(x)e^{A(x)}dx\right]e^{-A(x)} = \left[\int 2x^2e^{-x^3}dx\right]e^{x^3} = e^{x^3}\left[-\frac{2}{3}e^{-x^3}\right] = -\frac{2}{3}$$

Then, the general solution of the differential equation is: $y(x) = y_0 + y_p = ke^{x^3} - \frac{2}{3}$

Example 3.5: Solve the differential equation (E):

$$y' + 2y = \sin x$$

Answer: Let us find the solutions of (H.D.E)

$$y' + 2y = 0$$

Then, $$\frac{y'}{y} = -2$$

$$\int \frac{y'}{y} dx = \int -2dx = -2x + c$$

$$|y_0| = e^{-2x+c} = e^c \cdot e^{-2x}$$

$$y_0 = \pm e^c \cdot e^{-2x} = ke^{-2x}$$

Then, $y_0 = ke^{-2x}\ where, k \in \mathbb{R}$

- We use the method of variation of the constant, that is:

We note the function $k(x)$ such that $y_p = k(x)e^{-2x}$ is a particular solution of the equation (E).

$y'_p = k'(x)e^{-2x} - 2k(x)e^{-2x}$, we have: $k(x)e^{-2x} = y_p$

Then, $$y'_p = k'(x)e^{-2x} - 2y_p$$

$$y'_p + 2y_p = k'(x)e^{-2x} = \sin x$$

Then, $$k'(x) = e^{2x} \sin x$$

$$k(x) = \int e^{2x} \sin x\ dx = -\cos x\ e^{2x} + 2\int \cos x\ e^{2x}\ dx$$

$$\begin{cases} u(x) = e^{2x} \\ v'(x) = \sin x \end{cases} \begin{cases} u'(x) = 2e^{2x} \\ v(x) = -\cos x \end{cases}$$

The second case:

$$\begin{cases} u(x) = e^{2x} \\ v'(x) = \cos x \end{cases} \begin{cases} u'(x) = 2e^{2x} \\ v(x) = \sin x \end{cases}$$

$$\int e^{2x} \sin x\ dx = -\cos x\ e^{2x} + 2\sin x\, e^{2x} - 4\int e^{2x} \sin x\ dx$$

$$\int e^{2x} \sin x\ dx + 4\int e^{2x} \sin x\ dx = -\cos x\ e^{2x} + 2\sin x\, e^{2x}$$

$$5\int e^{2x} \sin x\ dx = -\cos x\ e^{2x} + 2\sin x\, e^{2x}$$

$$k(x) = \int e^{2x} \sin x\ dx = -\frac{1}{5}\cos x\ e^{2x} + \frac{2}{5}\sin x\, e^{2x}$$

The results of y_p is:

$$y_p(x) = \left(-\frac{1}{5}\cos x\, e^{2x} + \frac{2}{5}\sin x\ e^{2x}\right) e^{-2x}$$

$$y_p(x) = -\frac{1}{5}\cos x + \frac{2}{5}\sin x$$

We conclude the general solution $y(x)$:

$$y(x) = y_0 + y_p = ke^{-2x} - \frac{1}{5}\cos x + \frac{2}{5}\sin x$$

3.3.3. Cauchy–Lipschitz Theorem:

Let (E): $y' + a(x)y = b(x)$ be a linear differential equation. Then, for every $\alpha \in I$ and $\beta \in \mathbb{R}$, there exists a unique solution y of (E) such that $y(\alpha) = \beta$

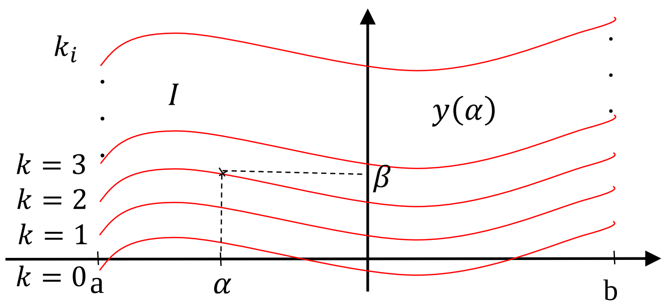

Fig 3.1: Curve of the solution $y = f(x)$.

Example 3.6: We consider (E): $x^2y' + 2xy = 5$

Solve (E) on the interval $]0, +\infty[$

Find the solution on this interval that also satisfies $y(1) = 6$.

Answer: We have: $y' + \frac{2}{x}y = \frac{5}{x^2}$

We have : $a(x) = \frac{2}{x}$ then, $A(x) = 2\ln x = \ln x^2$

$$y_0 = ke^{-\ln x^2} = \frac{k}{x^2}$$

$$y_p = e^{-A(x)}\left[\int b(x)e^{A(x)}dx\right]$$

$$y_p = e^{-\ln x^2}\left[\int \frac{5}{x^2}e^{\ln x^2}dx\right] = \frac{1}{x^2}\cdot[5x] = \frac{5}{x}$$

We conclude $y(x) = y_0 + y_p = \frac{k}{x^2} + \frac{5}{x}$, where: $k \in \mathbb{R}$.

- We have : $y(1) = 6$ then, $k + 5 = 6$, $k = 1$.

Then, $$y(x) = \frac{1}{x^2} + \frac{5}{x}$$

3.4. Second-order linear differential equation with constant coefficients:

3.4.1. Definition:

A second-order linear differential equation with constant coefficients is an equation of the form:

(E) : $ay'' + by' + cy = g(x)$

$a, b, c \in \mathbb{R}$ and $g(x)$: Continuous function on an interval, $I \subset \mathbb{R}$, and is called the *non-homogeneous term* (with $(a \neq 0)$)

The equation: (E_0) : $ay'' + by' + cy = 0$ is called the homogeneous equation associated to (E).

$$a(y_0'' + y_p'') + b(y_0' + y_p') + c(y_0 + y_p) = g(x)$$

$$\underbrace{ay_0'' + by_0' + cy_0}_{solution\ (E_0)} + \underbrace{ay_p'' + by_p' + cy_p}_{particular\ solution} = 0 + g(x)$$

Then, the general solution is:

$$y(x) = y_0(x) + y_p(x)$$

3.4.2. The solutions of the homogeneous equation:

The homogeneous equation is called: (E_0) : $ay'' + by' + cy = 0$

The equation: $ar^2 + br + c = 0$ is called the characteristic equation (C.E.) associated to (E). The discriminant is:

$$\Delta = b^2 - 4ac$$

Value of Δ	The solution of $y_0(x)$	The bases
$\Delta > 0$, $r_1, r_2 \in \mathbb{R}$	$y_0(x) = k_1 e^{r_1 x} + k_2 e^{r_2 x}$	$(e^{r_1 x}, e^{r_2 x})$
$\Delta = 0$, $r_0 \in \mathbb{R}$	$y_0(x) = (k_1 x + k_2) e^{r_0 x}$	$(xe^{r_0 x}, e^{r_0 x})$
$\Delta < 0$, $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$	$y_0(x) = e^{\alpha x}[k_1 \cos(\beta x) + k_2 \sin(\beta x)]$	$(e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x))$

Example 3.7: solve the differential equation to $\mathbb{R}$:

$$(E_0):\ 2y'' - 2y' + y = 0$$

Answer: The characteristic Equation: $2r^2 - 2r + 1 = 0$

$$\Delta = (-2)^2 - 4(2)(1) = 4 - 8 = -4 = (2i)^2$$

Then, the characteristic equation has two complex roots:

Then, the C.E. has two complex roots: $\begin{cases} r_1 = \frac{1}{2} + i\frac{1}{2} \\ r_2 = \frac{1}{2} - i\frac{1}{2} \end{cases}$

Then, $y_0(x) = e^{\frac{1}{2}x}\left(k_1 \cos\left(\frac{x}{2}\right) + k_2 \sin\left(\frac{x}{2}\right)\right)$, where: $k_1, k_2 \in \mathbb{R}$.

3.4.3. The particular solution of (E): $ay'' + by' + cy = g(x)$

A. Method of variation of constants

We consider the constants k_1 and k_2 in y_0 as unknown functions and look for a particular solution of (E) of the form:

$$y_p(x) = k_1(x)y_1(x) + k_2(x)y_2(x)$$

With the function $k_1(x)$ and $k_2(x)$ satisfying:

$$\begin{cases} k_1'(x)y_1(x) + k_2'(x)y_2(x) = 0 \\ k_1'(x)y_1'(x) + k_2'(x)y_2'(x) = \dfrac{g(x)}{a} \end{cases}$$

With,

$$y_0(x) = k_1y_1(x) + k_2y_2(x)$$

Example 3.8: solve the following equation (E) to $\mathbb{R}$

(E) : $y'' - y - 2y = \cos x$

Answer: lets us find the solution of (E_0): $y'' - y - 2y = 0$

C.E. : $r^2 - r - 2 = 0$, $\Delta = (-1)^2 - 4(-2)(1) = 9$

$$\begin{cases} r_1 = 2 \\ r_2 = -1 \end{cases}$$

With, $y_0(x) = k_1e^{2x} + k_2e^{-x}$, where: $k_1 , k_2 \in \mathbb{R}$

The particular solution:

We are looking for the functions k_1 and k_2 such that:

$$y_p(x) = k_1(x)e^{2x} + k_2(x)e^{-x}$$

$$\begin{cases} k_1'(x)e^{2x} + k_2'(x)e^{-x} = 0 \dots\dots\dots\dots\dots (1) \\ 2k_1'(x)e^{2x} - k_2'(x)e^{-x} = \cos x \dots\dots\dots\dots (2) \end{cases}$$

If we add the equations: (1) + (2)

$$3k_1'(x)e^{2x} = \cos x$$

$$k_1'(x) = \frac{1}{3}\cos x\, e^{-2x}$$

Then, $k_1(x) = \frac{1}{3}\int \cos x\, e^{-2x}dx$

Compute the integral of $\int \cos x\, e^{-2x}\, dx$:

$$\begin{cases} u = e^{-2x} , u' = -2e^{-2x} \\ v' = \cos x \ \ , v = \sin x \end{cases}$$

$$\int \cos x\, e^{-2x}\, dx = e^{-2x}\sin x + 2\int \sin x\, e^{-2x}\, dx$$

$$\begin{cases} u = e^{-2x} , u' = -2e^{-2x} \\ v' = \sin x \ \ , v = -\cos x \end{cases}$$

$$\int \cos x\, e^{-2x}\, dx = e^{-2x}\sin x - 2e^{-2x}\cos x - 4\int e^{-2x}\cos x\ dx$$

$$\int \cos x\, e^{-2x}\, dx + 4\int e^{-2x} \cos x\ \ dx = e^{-2x} \sin x - 2e^{-2x} \cos x$$

$$5\int \cos x\, e^{-2x}\, dx = e^{-2x} \sin x - 2e^{-2x} \cos x$$

$$\int \cos x\, e^{-2x}\, dx = \frac{1}{5} e^{-2x} \sin x - \frac{2}{5} e^{-2x} \cos x$$

$$k_1(x) = \frac{1}{3}\int \cos x\, e^{-2x} dx = \frac{1}{3}\left[\frac{1}{5} e^{-2x} \sin x - \frac{2}{5} e^{-2x} \cos x\right]$$

$$k_1(x) = \frac{1}{15} e^{-2x} \sin x - \frac{2}{15} e^{-2x} \cos x$$

We multiply the equation (1) by (-2) and add to equation (2): $(-2) * (1) + (2)$

$$-3k_2'(x)e^{-x} = \cos x$$

$$k_2'(x) = \frac{-1}{3} e^x \cos x$$

$$k_2(x) = \frac{-1}{3}\int e^x \cos x\, dx$$

Compute the integral $\int e^x \cos x\, dx$:

$$\begin{cases} u = e^x & , \quad u' = e^x \\ v' = \cos x & , \quad v = \sin x \end{cases}$$

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$$

$$\begin{cases} u = e^x & , \quad u' = e^x \\ v' = \sin x & , \quad v = -\cos x \end{cases}$$

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$\int e^x \cos x\, dx + \int e^x \cos x\, dx = e^x \sin x + e^x \cos x$$

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x\, dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x$$

$$k_2(x) = \frac{-1}{3}\int e^x \cos x\, dx = \frac{-1}{3}\left[\frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x\right]$$

$$k_2(x) = -\frac{1}{6} e^x \sin x - \frac{1}{6} e^x \cos x$$

$$y_p(x) = k_1(x)e^{2x} + k_2(x)e^{-x}$$

$$y_p(x) = \left[\frac{1}{15} e^{-2x} \sin x - \frac{2}{15} e^{-2x} \cos x\right] e^{2x} + \left[-\frac{1}{6} e^x \sin x - \frac{1}{6} e^x \cos x\right] e^{-x}$$

$$y_p(x) = \frac{1}{15}\sin x - \frac{2}{15}\cos x - \frac{1}{6}\sin x - \frac{1}{6}\cos x$$
$$y_p(x) = -\frac{1}{10}\sin x - \frac{3}{10}\cos x$$

Finally the general solution of (E) is:

$$y(x) = k_1 e^{2x} + k_2 e^{-x} - \frac{1}{10}\sin x - \frac{3}{10}\cos x$$

Where, $k_1, k_2 \in \mathbb{R}$

3.4.4. Special functions of the second-order differential equation:

Case 1: $ay'' + by' + cy = e^{mx}P_n(x)$		
If ***m* is not a root** of the characteristic equation	If ***m* is a simple root** of the characteristic equation	If ***m* is a double root** of the characteristic equation
$y_p = e^{mx}Q_n(x)$	$y_p = xe^{mx}Q_n(x)$	$y_p = x^2e^{mx}Q_n(x)$
Example 3.9: $y'' + y' - 2y = e^{3x}$ H.D.E :$y'' + y' - 2y = 0$ C.E.: $r^2 + r - 2 = 0$ $\Delta = 9, \sqrt{\Delta} = 3, \begin{cases} r_1 = -2 \\ r_2 = 1 \end{cases}$ $m = 3$ is not a root of (C.E.). $y_0(x) = k_1e^{-2x} + k_2e^x$ $k_1, k_2 \in \mathbb{R}$ The particular solution: $y_p(x) = Ae^{3x}$, $y_p'(x) = 3Ae^{3x}$ $y_p''(x) = 9Ae^{3x}$ $9Ae^{3x} + 3Ae^{3x} - 2Ae^{3x} = e^{3x}$ $10Ae^{3x} = e^{3x}, A = \frac{1}{10}$ $y_p(x) = \frac{1}{10}e^{3x}$ $y(x) = k_1e^{-2x} + k_2e^x + \frac{1}{10}e^{3x}$	**Example 3.10:** $y'' - 3y' + 2y = e^x$ H.D.E :$y'' - 3y' + 2y = 0$ C.E.: $r^2 - 3r + 2 = 0$ $\Delta = 1, \sqrt{\Delta} = 1, \begin{cases} r_1 = 1 \\ r_2 = 2 \end{cases}$ $m = 1$ is a simple root of C.E. $y_0(x) = k_1e^x + k_2e^{2x}$ $k_1, k_2 \in \mathbb{R}$ The particular solution: $y_p(x) = Axe^x$ $y_p'(x) = Ae^x + Axe^x$ $y_p''(x) = 2Ae^x + Axe^x$ $2Ae^x + Axe^x - 3Ae^x - 3Axe^x + 2Axe^x = e^x$ $Ae^x(-1 + 3x - 3x) = e^x$ $-Ae^x = e^x, A = -1$ $y_p(x) = -xe^x$ $y(x) = k_1e^x + k_2e^{2x} - xe^x$	**Example 3.11:** $y'' - 2y' + y = e^x$ H.D.E :$y'' - 2y' + y = 0$ C.E.: $r^2 - 2r + 1 = 0$ $(r-1)^2 = 0, r = 1$ double root. $m = 1$ is a double root of C.E. $y_0(x) = (k_1 + k_2x)e^x$ $k_1, k_2 \in \mathbb{R}$ The particular solution: $y_p(x) = Ax^2e^x$, $y_p'(x) = 2Axe^x + Ax^2e^x$ $y_p''(x) = 2Ae^x + 4Axe^x + Ax^2e^x$ $y_p''(x) - 2y_p'(x) + y_p(x) = e^x$ $2Ae^x = e^x, A = \frac{1}{2}$ $y_p(x) = \frac{1}{2}x^2e^x$

<table>
<tr><td></td><td></td><td>$y(x) = (k_1 + k_2 x)e^x + \frac{1}{2}x^2 e^x$</td></tr>
<tr><td colspan="3">Case 2: $ay'' + by' + cy = e^{mx}P_n(x)\cos(\beta x)$</td></tr>
<tr><td colspan="2">$m + i\beta$ is not a root of the characteristic equation.</td><td>$m + i\beta$ a simple root of the characteristic equation.</td></tr>
<tr><td colspan="2">$y_p = e^{mx}(A\cos(\beta x) + B\sin(\beta x))$</td><td>$y_p = xe^{mx}(A\cos(\beta x) + B\sin(\beta x))$</td></tr>
<tr><td colspan="2">Example 3.12:
$y'' + y = e^{0x}\cos(2x)$
$m = 0, \beta = 2, m + i\beta = 0 + 2i$
H.D.E : $y'' + y = 0$
C.E : $r^2 + 1 = 0, r = \pm i,$
$m + i\beta$ is not a root of C.E.
$y_0(x) = (k_1\cos x + k_2\sin x)$
The particular solution :
$y_p(x) = (A\cos(2x) + B\sin(2x))$
$y_p'(x) = (-2A\sin(2x) + 2B\cos(2x))$
$y_p''(x) = (-4A\cos(2x) - 4B\sin(2x))$
$y_p''(x) + y_p(x) = \cos(2x)$
$(-4A\cos(2x) - 4B\sin(2x)) + (A\cos(2x) + B\sin(2x)) = \cos(2x)$
$-3A\cos(2x) - 3B\sin(2x) = \cos(2x)$
$\begin{cases} -3A = 1 \\ -3B = 0 \end{cases}, \begin{cases} A = \frac{-1}{3} \\ B = 0 \end{cases}$
$y_p(x) = -\frac{1}{3}\cos(2x)$
$y(x) = k_1\cos x + k_2\sin x - \frac{1}{3}\cos(2x)$</td><td>Example 3.13:
$y'' + 4y = e^{0x}\cos(2x)$
$m = 0, \beta = 2, m + i\beta = 0 + 2i$
H.D.E : $y'' + 4y = 0$
C.E : $r^2 + 4 = 0, r = \pm 2i,$
$m + i\beta$ a simple root of C.E.
$y_0(x) = (k_1\cos(2x) + k_2\sin(2x))$
The particular solution :
$y_p(x) = x(A\cos(2x) + B\sin(2x))$
$y_p'(x) = A\cos(2x) + B\sin(2x) + x(-2A\sin(2x) + 2B\cos(2x))$
$y_p''(x) = -4A\sin(2x) + 4B\cos(2x) + x(-4A\cos(2x) - 4B\sin(2x))$
$y_p''(x) + 4y_p(x) = \cos(2x)$
$-4A\sin(2x) + 4B\cos(2x) = \cos(2x)$
$\begin{cases} -4A = 0 \\ 4B = 1 \end{cases}, \begin{cases} A = 0 \\ B = \frac{1}{4} \end{cases}$
$y_p(x) = \frac{1}{4}\cos(2x)$
$y(x) = k_1\cos(2x) + k_2\sin(2x) + \frac{1}{4}\cos(2x)$</td></tr>
</table>

3.5. Partial Differential Equations

Let $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of two real variables defined in the neighborhood of a point $A(a, b)$. If the function $x \to f(x, b)$ has a derivative with respect to x, we denote it by ${f_x}'(a, b)$.

We can define the partial derivative with respect to x by:

$$f_x' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x, y) \to f_x'(x, y)$$

Similarly, the partial derivative with respect to y is defined in the same way.

We denote:

$$f_x' = \frac{\partial f}{\partial x}, f_y' = \frac{\partial f}{\partial y}$$

3.5.1. Successive Derivatives

In the same way, we can define the derivative of the function f_x' with respect to x, denoted by f_{x^2}'' or $\frac{\partial^2 f}{\partial x^2}$.

Also, the derivative of the function f_x' with respect to y is denoted by f_{xy}'' or $\frac{\partial^2 f}{\partial y \partial x}$.

The derivative of the function f_y' with respect to y is denoted by f_{y^2}'' or $\frac{\partial^2 f}{\partial y^2}$.

The derivative of the function f_y' with respect to x is denoted by f_{yx}'' or $\frac{\partial^2 f}{\partial x \partial y}$.

3.5.2. Schwarz's Theorem

If the partial derivatives f_{xy}'' and f_{yx}'' are continuous, then these derivatives are equal:

$$f_{xy}'' = f_{yx}''$$

Example 3.12: Compute :

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$$

For the function:

$$f(x, y) = x^2 + xy + y^2$$

Answer:

$$\frac{\partial f}{\partial x} = 2x + y, \frac{\partial^2 f}{\partial x^2} = 2, \frac{\partial^2 f}{\partial y \partial x} = 1$$

$$\frac{\partial f}{\partial y} = 2y + x, \frac{\partial^2 f}{\partial y^2} = 2, \frac{\partial^2 f}{\partial x \partial y} = 1$$

3.5.3. Derivatives of a Function Composed of Two Variables

Let the function $F(x, y) = f(u, v)$, where u and v are functions of x and y that admit partial derivatives.

If f itself has partial derivatives, then F also has partial derivatives, and we have:

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial v}$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial f}{\partial v}$$

Example 3.13: Let $F(x, y) = f(u, v)$**,** where: $u = x + y, v = x - y$ and $f(u, v) = u^2 + v^2$

Determine the Derivatives of a Function $F(x, y)$?

Answer:

Method 1: compute $F(x, y)$ in terms of x and y:

$$F(x, y) = (x + y)^2 + (x - y)^2 = 2x^2 + 2y^2$$

Compute the partial derivatives directly:

$$\frac{\partial F}{\partial x} = 4x, \frac{\partial F}{\partial y} = 4y$$

Method 2: verify using the chain rule according to the formula:

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial v}$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial f}{\partial v}$$

Now:

$$\frac{\partial f}{\partial u} = 2u, \frac{\partial f}{\partial v} = 2v$$

And

$$\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = 1, \frac{\partial u}{\partial y} = 1, \frac{\partial v}{\partial y} = -1$$

Then,

$$\frac{\partial F}{\partial x} = (1)(2u) + (1)(2v) = 2(u + v)$$

$$\frac{\partial F}{\partial y} = (1)(2u) + (-1)(2v) = 2(u - v)$$

Substitute $u = x + y, v = x - y$**:**

$$\frac{\partial F}{\partial x} = 2(u + v) = 2(x + y + x - y) = 4x$$

$$\frac{\partial F}{\partial y} = 2(u - v) = 2(x + y - x + y) = 4y$$

3.5.4. Total Differentials

Let $U: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of two variables that has continuous partial derivatives.

The **total differential** of U is written as:

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

If $U(x, y) = constant$, then $dU = 0$.

Example 3.14: let's consider the function.

$$U(x, y) = x^2 + y^2$$

Compute the partial derivatives

$$\frac{\partial U}{\partial x} = 2x, \quad \frac{\partial U}{\partial y} = 2y$$

Write the total differential

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

$$dU = 2xdx + 2ydy$$

If $U(x, y) = constant$

That means $x^2 + y^2 = C$, which is the equation of a circle.

Since U is constant along this curve:

$$dU = 0$$

So:

$$2xdx + 2ydy = 0$$

$$xdx + ydy = 0$$

3.5.5. Exact Total Differential Forms

Consider the following differential equation:

$$M(x, y)dx + N(x, y)dy = 0$$

If we can find a function $U: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:

$$\frac{\partial U}{\partial x}(x, y) = M(x, y), \frac{\partial U}{\partial y}(x, y) = N(x, y)$$

Then the solution is given by:

$$U(x, y) = Constant$$

3.5.6. Application to the Integration of First-Order Differential Equations

Consider the following differential equation:

$$M(x, y) + N(x, y)y' = 0$$

This equation can be written in the form:

$$M(x, y)dx + N(x, y)dy = 0$$

if we can find a function $U: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:

$$\frac{\partial U}{\partial x}(x, y) = M(x, y), \frac{\partial U}{\partial y}(x, y) = N(x, y)$$

Then the solution is given by:

$$U(x, y) = Constant$$

Example 3.15:

Solve the differential equation

$$2xydx + (x^2 + 3y^2)dy = 0$$

Write it as $M(x, y)dx + N(x, y)dy = 0$

$$M(x, y) = 2xy\,, \qquad N(x, y) = x^2 + 3y^2$$

Compute the partial derivatives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x, \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3y^2) = 2x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the form is exact on $\mathbb{R}^2$.

We seek U such that

$$\frac{\partial U}{\partial x} = M = 2xy, \frac{\partial U}{\partial y} = N = x^2 + 3y^2$$

Integrate M with respect to x:

$$U(x, y) = \int 2xydx = x^2y + \phi(y)$$

Where $\phi(y)$ is an arbitrary function of y.

Differentiate this U with respect to y and match N:

$$\frac{\partial U}{\partial y} = x^2 + \phi'(y) = x^2 + 3y^2$$

Thus $\phi'(y) = 3y^2$, so $\phi(y) = y^3 + C_0$

Therefore

$$U(x, y) = x^2y + y^3 + C_0$$

The solution of the differential equation is level set of U:

$$x^2y + y^3 = C$$

Where C is an arbitrary constant (absorbing C_0)

Chapter 4: Series

4.1. Numerical series

4.1.1. Generalities

The numerical series $(U_n)_{n\in\mathbb{N}}$ is a sequence of terms extending to infinity, denoted by:

$$\sum_{n=0}^{+\infty} U_n$$

The partial sum S_N of the first (N+1) terms is:

$$S_N = \sum_{n=0}^{N} U_n = U_0 + U_1 + U_2 + \cdots + U_N$$

We say that the series converges if the sequence of its partial sums (S_N) has a finite limit as (N → ∞), and the result is called the sum of the series (S).

We say that the series diverges if the limit does not exist or is infinite.

The remainder of a convergent series is given by:

$$R_N = S - S_N = \sum_{n=N+1}^{+\infty} U_n$$

Example 4.1: Let the following geometric series be:

$$\sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n$$

The partial sum is:

$$S_N = \sum_{n=0}^{N} \left(\frac{1}{2}\right)^n = \frac{1-\left(\frac{1}{2}\right)^{N+1}}{1-\frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^N$$

When N → +∞, S_n → 2, then the series converges and its sum is S = 2.

The remainder is:

$$R_N = 2 - S_N = \left(\frac{1}{2}\right)^N$$

4.1.2. Necessary condition for convergence

If a series $\sum U_n$ converges, the general term U_n must tend to 0 as n → +∞. This is a necessary but not sufficient condition.

Example 4.2: Let the following harmonic series:

$$\sum_{n=1}^{+\infty} \frac{1}{n}$$

The general term $U_n = \frac{1}{n}$ does tend to 0 as $n \to +\infty$.

However, the series diverges, so the necessary condition is not sufficient.

4.1.3. Properties of Convergent Numerical Series

If $\sum U_n$ and $\sum V_n$ are convergent series, and λ is a constant:

- $\sum U_n + V_n$ converges and $\sum U_n + V_n = \sum U_n + \sum V_n$
- $\sum \lambda U_n$ converges and $\sum \lambda U_n = \lambda \sum U_n$
- The nature of a series does not change if a finite number of its terms are modified.

Example 4.3: Let the following series:

$$\sum_{n=1}^{+\infty} \left(\frac{1}{2^n} + \frac{1}{3^n}\right)$$

We know that $\sum_{n=1}^{+\infty} \frac{1}{2^n}$ *converges* $(S = 1)$ *and* $\sum_{n=1}^{+\infty} \frac{1}{3^n}$ converges $(S = \frac{1}{2})$.

Therefore, the sum of the two series also converges, and its total sum is $S = 1 + \frac{1}{2}$

4.1.4. Positive-term numerical series

A series $\sum U_n$ with positive terms means that all terms U_n are positive or zero.

A positive-term series is said to **converge if and only if** the sequence of its partial sums is bounded.

4.1.5. Convergence Criteria

Comparison Test:

If $0 \leq U_n \leq V_n$ for n sufficiently large, then:

- If $\sum V_n$ converges, then $\sum U_n$also converges.
- If $\sum U_n$ diverges, then $\sum V_n$ also diverges.

Example 4.4: Let us study the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

For all $n \geq 1$, we have $0 < \frac{1}{n^2+1} \leq \frac{1}{n^2}$.

The Riemann series $\sum \frac{1}{n^2}$ is a known convergent reference series.

Therefore, by the comparison test, $\sum \frac{1}{n^2+1}$ converges.

4.1.6. Equivalence Rule

If $U_n > 0$ and $V_n > 0$ and $U_n \sim V_n$ (that is, $\lim_{n\to\infty} \frac{U_n}{V_n} = L \in \mathbb{R}^*$), then the series $\sum U_n$ and $\sum V_n$ have the same behavior (i.e., they are either both convergent or both divergent).

Example 4.4: Let us study the convergence of $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$.

As $n \to \infty$,, we have $\sin\left(\frac{1}{n}\right) \sim \frac{1}{n}$

The harmonic series $\sum \frac{1}{n}$ diverges.

Therefore, by the equivalence rule, $\sum \sin\left(\frac{1}{n}\right)$ diverges.

4.1.7. D'Alembert's Rule (Ratio Test)

For a series with strictly positive terms:

$$\lim_{n\to+\infty} \frac{U_n + 1}{U_n} = L$$

- If ($L < 1$), the series converges.
- If ($L > 1$), the series diverges.
- If ($L = 1$), the test is inconclusive.

Example 4.5: Let us study the convergence of $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

We have: $U_n = \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)n!}{(n+1)(n+1)^n} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1$$

The series is converge.

4.1.8. Cauchy's rules

For a series with positive terms:

$$\lim_{n\to+\infty} \sqrt[n]{U_n} = L$$

- If ($L < 1$), the series converges.
- If ($L > 1$), the series diverges.
- If ($L = 1$), the test is inconclusive.

Example 4.6: Let us study the convergence of $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

- $U_n = \left(\frac{n+1}{2n+1}\right)^n$
- $\sqrt[n]{U_n} = \frac{n+1}{2n+1}$
- The limit is $\lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2}$, where $\frac{1}{2} < 1$, the series is **converge**.

4.1.9. Cauchy's Integral Test

The integral test for the numerical series $\sum U_n$ is based on the following formula:

$$\sum_{n=1}^{\infty} U_n = \int_1^{+\infty} f(x)\,dx$$

Example 4.7: Let us study the convergence of the Riemann series $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \int_1^{+\infty} \frac{1}{x}\,dx$$

$$\int_1^{+\infty} \frac{1}{x}\,dx = [\ln x]_1^{+\infty} = \lim_{x\to+\infty} \ln x - \ln 1 = +\infty$$

Therefore, we conclude that the series diverges.

4.1.10. Series with General Terms

A. Alternating Series

An alternating series is a series whose terms change sign alternately, of the form $\sum(-1)^n a_n$ with $a_n \geq 0$.

Leibniz's Theorem: If the sequence (a_n) is decreasing and tends to 0, then the alternating series $\sum(-1)^n a_n$ converges.

Example 4.8: The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

- Here, $a_n = \frac{1}{n}$.
- The sequence (a_n) is indeed positive, decreasing, and $\lim_{n\to\infty} \frac{1}{n} = 0$.
- According to Leibniz's theorem, the series converges.

B. Absolutely Convergent Series

A series $\sum U_n$ is said to be **absolutely convergent** if the series of absolute values $\sum|U_n|$ converges

Theorem: If a series is absolutely convergent, then it is convergent (CVA ⇒ CVS). The converse is false.

Example 4.9: The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

- The series of absolute values is $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$.
- It is a Riemann series with $p = 2 > 1$, it converge.
- Therefore, the original series is absolutely convergent and consequently, it converges.

C. Conditionally Convergent Series

A series is said to be conditionally convergent if it is convergent but not absolutely convergent.

Example 4.10: The alternating harmonic series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

- We have seen that it converges (according to Leibniz's criterion).
- The series of absolute values is $\sum \left|\frac{(-1)^{n+1}}{n}\right| = \sum \frac{1}{n}$, which is the harmonic series.
- The harmonic series diverges.
- Since the series converges but not absolutely, it is conditionally convergent**.**

D. Abel's Criterion (First Abel Test for Series)

Let $\sum a_n b_n$ be a series. If:

- The sequence (a_n) is positive, decreasing, and tends to 0.
- The sequence of partial sums of $\sum b_n$ is bounded. Then the series $\sum a_n b_n$ converges.

Example 4.11: Let us study the convergence of $\sum_{n=1}^{\infty} \frac{\cos n}{n}$

- Let $a_n = \frac{1}{n}$ and $b_n = \cos n$.
- The sequence $a_n = \frac{1}{n}$ is positive, decreasing, and tends to 0.
- We need to verify whether the sequence of partial sums of $\sum \cos n$ is bounded. This is a known result in analysis, since $\sum_{k=1}^{N} \cos k$ is bounded by $\frac{1}{sin\left(\frac{1}{2}\right)}$
- Therefore, according to Abel's criterion, the series $\sum_{n=1}^{\infty} \frac{\cos n}{n}$ converges.

4.2. Power Series

4.2.1. Definition, Abel's Lemma, and Radius of Convergence

A power series is a series of the form $\sum_{n=0}^{\infty} a_n x^n$, where a_n is a sequence of coefficients and x is a variable.

A. Abel's Lemma:

If there exists a real number $x_0 \neq 0$ such that the sequence $(a_n x_0^n)$ is bounded, then the power series:

$$\sum a_n x^n$$

Converges absolutely for all $|x| < |x_0|$.

B. Radius of Convergence (R):

It is the largest positive real number such that the series

$$\sum a_n x^n$$

Converges for all $|x| < R$. The set of values of x for which the series converges is called the *interval of convergence*, which is at least $]-R, R[$.

Convergence at the endpoints $x = R$ and $x = -R$ must be studied separately.

C. Determination of the Radius of Convergence

One can use D'Alembert's or Cauchy's tests. For example, if the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

Exists, then the radius of convergence is given by $R = 1/L$.

If $L = 0$, then $R = \infty$.

if $L = \infty$, then $R = 0$.

D. Hadamard's Formula

The radius of convergence R is given by

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

If the limit exists, it is equal to

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

Example 4.12: Consider the power series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

- We have: $a_n = \frac{1}{n!}$
- We use the formula of Alembert :

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

Since $L = 0$, the radius of convergence is $R = \infty$.

The series therefore converges for all $x \in R$.

4.2.2. Properties of Power Series

A. Linearity and Product

If $\sum a_n x^n$and $\sum b_n x^n$ converge with radii of convergence R_a and R_b, then $\sum(a_n + b_n)x^n$ converges with a radius $R \geq min(R_a, R_b)$.

The Cauchy product of two convergent power series is also a power series whose radius of convergence is at least $min(R_a, R_b)$.

B. Normal Convergence

A power series

$$\sum a_n x^n$$

converges *normally* on every closed interval $[a, b]$ contained within the open interval of convergence $]-R, R[$.

C. Continuity

The sum of a power series is a continuous function on its open interval of convergence.

D. Term-by-Term Integration and Differentiation

On the open interval of convergence $]-R, R[$, a power series can be integrated and differentiated term by term, and the resulting series have the same radius of convergence.

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\int \left(\sum_{n=0}^{\infty} a_n x^n\right) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Example 4.13: The Geometric Series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$

- Its radius of convergence is $R = 1$.
- By differentiating term by term, we obtain: $\sum_{n=1}^{\infty} n x^{n-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$. for $|x| < 1$
- By integrating term by term, we obtain: $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \int \frac{1}{1-x} dx = -\ln(1-x)$, for $|x| < 1$.

4.3. Expansion in Power Series

4.3.1. Function Expandable into a Power Series

A function $f(x)$ is said to be *expandable into a power series* on an interval I if it can be written in the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all $x \in I$.

A. Taylor–Maclaurin Series

If a function f can be expanded into a power series in a neighborhood of 0, then its coefficients are given by

$$a_n = \frac{f^{(n)}(0)}{n!}$$

The resulting power series is called the Maclaurin series of the function f.

Maclaurin series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'^{(0)} x + \frac{f''(0)}{2!} x^2 + \cdots$

Uniqueness of the Expansion: If a function f can be expanded into a power series, its expansion is unique.

Example 4.14: Let's expand $f(x) = e^x$ in a power series about 0 (Maclaurin series).

Coefficients by derivatives at 0. For all $n \geq 0$,

$$f^{(n)}(x) = e^x \ \Rightarrow f^{(n)}(0) = 1$$

Hence the Maclaurin coefficients are:

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

Series representation.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Radius of convergence. Using the ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

So $R = \frac{1}{0} = \infty$, Thus the series converges for every real x (indeed for every complex x).

B. Useful properties (on ℝ or ℂ):

Term-by-term differentiation: $\frac{d}{dx} e^x = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

Term-by-term integration: $\int_0^x e^t \, dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = e^x - 1$

4.3.2. Applications

A. Expansions of Common Functions

Many standard functions have well-known power series expansions, which are fundamental tools in mathematics and physics.

- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$for all $x \in R$.
- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$for all $x \in R$.
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ with $|\ x\ | < 1$.

B. Solving Differential Equations

One can look for a solution of a differential equation in the form of a power series $\sum a_n x^n$ By substituting the series and its derivatives into the equation, a recurrence relation for the coefficients ana_nan is obtained, which allows them to be determined.

Example 4.15: Find the solution of (y' = y) in the form of a power series.

Suppose that $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then, $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

The equation $y' = y$ becomes $\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n$

By the uniqueness of the coefficients, we have $(n+1)a_{n+1} = a_n$, let $a_{n+1} = \frac{a_n}{n+1}$

Starting from a_0, we find $a_1 = a_0, a_2 = \frac{a_1}{2} = \frac{a_0}{2}, a_3 = \frac{a_2}{3} = \frac{a_0}{3.2} = \frac{a_0}{3!}$, etc.

By induction, we have $a_n = \frac{a_0}{n!}$

The solution is $y(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$

This is indeed the general solution of the differential equation.

4.4. Fourier series

4.4.1. Definition

A trigonometric series (or Fourier series) is any series of functions of the form:

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos x_n + b_n \sin x_n)$$

a_0, a_n and b_n are the coefficients of the Fourier series.

f is periodic with period T if:

$f(x+T) = f(x)$ $\qquad (T > 0)$

$\sin x$: is periodic with period 2π.

$\cos x$: is periodic with period 2π.

$\tan x = \frac{\sin x}{\cos x}$: is periodic with period π.

Let f be a periodic function with period T:

We set $T = 2L$, hence $L = \frac{T}{2}$.

The Fourier series associated with f is:

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(a_n \cos \frac{nx\pi}{L} + b_n \sin \frac{nx\pi}{L}\right)$$

With,

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

4.4.2. Dirichlet's Theorem

Let f a function where f is continuous on $[a,b]$, f is piecewise differentiable on same interval, and for all $x_0 \in [a,b]$, $f({x_0}^+)$ and $f({x_0}^-)$ are finite and exist.

Then, the Fourier series of f converges to:

Therefore, f is discontinuous at x.

$$\begin{cases} f(x)\ if\ f\ is\ continuous\ on\ x\ (S_f(x)=f(x)) \\ Else\ \dfrac{f(x_0{}^+)+f(x_0{}^-)}{2}\ \ therefore\ f\ is\ discontinuous\ at\ \left(S_f(x)=\dfrac{f(x_0{}^+)+f(x_0{}^-)}{2}\right) \end{cases}$$

Note: if f is even ($f(-x)=f(x)$) then,

$$\begin{cases} b_n=0 \quad \forall\, n\in\mathbb{N}^* \\ a_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx=\frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \end{cases}$$

If f is odd ($f(-x)=-f(x)$), then,

$$\begin{cases} a_n=0 \quad \forall\, n\in\mathbb{N} \\ b_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx=\frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases}$$

$$\int \cos(\omega x)\,dx=\frac{1}{\omega}\sin(\omega x)$$

$$\int \sin(\omega x)\,dx=-\frac{1}{\omega}\cos(\omega x)$$

$$\sin n\pi=0$$

$$\cos n\pi=(-1)^n, \forall\, n\in\mathbb{N}$$

$$\sin\left(\theta\pm\frac{\pi}{2}\right)=\cos\theta$$

$$\cos\left(\theta\pm\frac{\pi}{2}\right)=\mp\sin\theta$$

Example 4.16: Let f be a function that is neither even nor odd, periodic with period 10, defined by:

$$f(x)=\begin{cases} 0\ \ if\ \ -5<x<0 \\ 3\ \ if\ \ \ 0<x<5 \end{cases}$$

Déterminer les coefficients de Fourier

Donner la période de Fourier associée à f

La réponse : on a f une fonction périodique de période 10 , T=10= 2L alors L=5.

Determine the Fourier coefficients.

Give the Fourier period associated with f.

Answer:

Since f is a periodic function with period 10, we have T=10=2L so that L=5.

Calculate of a_n ,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0} f(x)\cos\left(\frac{n\pi x}{5}\right)dx + \int_{0}^{5} f(x)\cos\left(\frac{n\pi x}{5}\right)dx\right]$$

$$= \frac{1}{5}\int_{0}^{5} 3\cos\left(\frac{n\pi x}{5}\right)dx = \frac{3}{5}\int_{0}^{5} \cos\left(\frac{n\pi x}{5}\right)dx = \frac{3}{5}\frac{5}{n\pi}\left[\sin\left(\frac{n\pi x}{5}\right)\right]_0^5$$

$$= \frac{3}{n\pi}[\sin(n\pi) - \sin(0)] = 0$$

Calculate of a_0

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{5}\int_{-5}^{5} f(x)dx = \frac{1}{5}\int_{0}^{5} 3dx = \frac{3}{5}[x]_0^5 = \frac{3}{5}\times 5 = 3$$

Calculate of b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{5}\right)dx = \frac{1}{5}\int_{0}^{5} 3\sin\left(\frac{n\pi x}{5}\right)dx = \frac{3}{5}\int_{0}^{5} \sin\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{3}{5}\times\left(\frac{-5}{n\pi}\right)\left[\cos\left(\frac{n\pi x}{5}\right)\right]_0^5 = \frac{-3}{n\pi}[\cos(n\pi) - \cos(0)] = \frac{-3}{n\pi}((-1)^n - 1)$$

$$= \frac{3((-1)^{n+1} + 1)}{n\pi}$$

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right) = \frac{3}{2} + \sum_{n=1}^{+\infty}\left(\frac{3((-1)^{n+1} + 1)}{n\pi}\sin\left(\frac{n\pi x}{5}\right)\right)$$

$$= \frac{3}{2} + \frac{3}{\pi}\sum_{n=1}^{+\infty}\left(\frac{((-1)^{n+1} + 1)}{n}\sin\left(\frac{n\pi x}{5}\right)\right)$$

4.4.3. General Rule

Let $f(x)$ be a periodic function with period $2L$. To determine a_n and b_n, we have:

$$a_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

With $c \in \mathbb{R}$

In the case of : $c = -L$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

4.4.4. Parseval's Equality

$$\frac{{a_0}^2}{2} + \sum_{n=1}^{+\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L}\big(f(x)\big)^2 dx$$

Chapter 5: Fourier Transform

5.1. Definition and Properties

5.1.1. Definition

The Fourier Transform is convert a function from the time domain $f(t)$ to the frequency domain, as follow:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, dt$$

where:

$F(\omega)$ is the Fourier transform (frequency-domain representation),

ω is the angular frequency (in radians per second),

The Inverse Fourier Transform allows us to recover $f(t)$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega$$

5.1.2. Differentiation in the time domain

$$\mathcal{F}\left(\frac{d^n f}{dt^n}\right)(t) = (i\omega)^n f(\omega)$$

5.1.3. Differentiation in the frequency domain

$$\frac{d^n f}{dt^n}(\omega) = \mathcal{F}((it\omega)^n f)(t)$$

5.1.4. Main Properties of the Fourier Transform

Linearity:

$$\mathcal{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega)$$

A. Time Shifting:

If , $f(t - t_0) \to e^{-i\omega t_0} F(\omega)$

B. Frequency Shifting:

If, $e^{i\omega_0 t} f(t) \to F(\omega - \omega_0)$

C. Scaling (Dilation):

If, $f(at) \to \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$

D. Convolution Theorem

$$\mathcal{F}\{f * g\} = F(\omega) \cdot G(\omega)$$

Example 5.1: Fourier Transform of an Exponential Function

Let $f(t) = e^{-a|t|}$ with $a > 0$, we have:

$$F(\omega) = \int_{-\infty}^{+\infty} e^{-a|t|} e^{-i\omega t}\, dt$$

Where $|t| = \begin{cases} -t, & t < 0 \\ t, & t > 0 \end{cases}$, we have:

$$F(\omega) = \int_{-\infty}^{0} e^{-a(-t)} e^{-i\omega t} dt + \int_{0}^{+\infty} e^{-at} e^{-i\omega t}\, dt$$

$$F(\omega) = \int_{-\infty}^{0} e^{(a-i\omega)t}\, dt + \int_{0}^{+\infty} e^{-(a+i\omega)t}\, dt = \frac{1}{a - i\omega} + \frac{1}{a + i\omega} = \frac{2a}{a^2 + \omega^2}$$

Then:

$$F(\omega) = \int_{-\infty}^{+\infty} e^{-a|t|} e^{-i\omega t}\, dt = \frac{2a}{a^2 + \omega^2}$$

5.2. Application to the Solution of Differential Equations

The Fourier Transform is a tool for solving linear differential equations,

$$\mathcal{F}\left(\frac{d^n f(t)}{dt^n}\right) = (2i\pi\omega)^n F(\omega)$$

Example:

$$\frac{d^2 f(t)}{dt^2} + f(t) = g(t), with\ g(t) = e^{-t}$$

$$\mathcal{F}\left[\frac{d^2 f(t)}{dt^2}\right] = (2i\pi\omega)^2 F(\omega) = -(2\pi\omega)^2 F(\omega)$$

Then,

$$-(2\pi\omega)^2 F(\omega) + F(\omega) = G(\omega)$$

$$F(\omega)(1 - (2\pi\omega)^2) = G(\omega)$$

$$F(\omega) = \frac{G(\omega)}{1 - (2\pi\omega)^2}$$

The Fourier Transform of $g(t) = e^{-t}$ is :

$$G(\omega) = \frac{1}{1 + 2\pi i\omega}$$

Then,

$$F(\omega) = \frac{1}{(1 + 2\pi i\omega)(1 - (2\pi\omega)^2)}$$

Chapter 6: Laplace Transform

6.1. Definition and Properties

6.1.1. Definition

The Laplace Transform of a function $f(t)$, defined for $t \geq 0$, is:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

Where s is a complex number $s = \sigma + i\omega$.

6.1.2. Properties

A. Linearity

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

B. Derivative in time domain

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$$

C. Integration in time domain

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

D. Shifting property

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Example 6.1:

$$f(t) = e^{2t}$$

$$F(s) = \int_0^\infty e^{2t}e^{-st}dt = \int_0^\infty e^{-(s-2)t}dt = \frac{1}{s-2}, \qquad for\ s > 2$$

6.2. Application to the solution of Differential Equations

The Laplace Transform is especially powerful for solving linear differential equations with initial conditions**.**

Example 6.2: Solve

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0, y(0) = 1, y'(0) = 0$$

Apply Laplace Transform

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 0$$

Using properties

$$(s^2Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) + 2Y(s) = 0$$

Substitute initial values $y(0) = 1, y'(0) = 0$

$$(s^2Y(s) - s) + 3(sY(s) - 1) + 2Y(s) = 0$$

$$Y(s)(s^2 + 3s + 2) - s - 3 = 0$$

$$Y(s)(s^2 + 3s + 2) = s + 3$$

$$Y(s) = \frac{s+3}{s^2 + 3s + 2} = \frac{s+3}{(s+1)(s+2)}$$

Partial Fraction Decomposition

$$Y(s) = \frac{2}{s+1} - \frac{1}{s+2}$$

Inverse Laplace Transform

$$y(t) = 2e^{-t} - e^{-2t}$$

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