Mathematics Course – SP Geology – Complex numbers

E-mail: m.azioune@centre-univ-mila.dz

Introductory note

Program according to sections:
- graphic representation, operations, conjugate, module, argument, trigonometric form: all sections

Prerequisites

Trigonometric circle – second degree polynomials – trigonometry – exponential – absolute value

Course Map

1. All complex numbers
2. Second degree polynomials
3. Module and argument
4. Exponential notation
5. Characterization of pure real and imaginary

1. All complex numbers

Definition :

- The set of complex numbers, denoted C, is a set of numbers defined by the following properties:
- C contains R the set of real numbers
- the calculation rules in C (addition and subtraction, multiplication and division) are the same as in R
- there exists in C a number i such that $i^2 = -1$
- a complex number z can be written uniquely in the form $z = a + ib$ **with real a and b.**

i corresponds to an "invented" number: $\sqrt{-1}$.

In the set of complex numbers, a square is no longer necessarily positive, as is the case in the set of real numbers.

The solution of the equation $z^2 = -1$
herefore has two solutions in the set of complexes (while in the set of real numbers, it has no solution): $z_1 = i$ et $z_2 = -i$

Hence: the solution of the equation $z^2 = k$ with real **k < 0** has two solutions in the set of complexes (while in the set of real numbers, it has no solution):

$z_1 = i\sqrt{-k}$ and
$z_2 = -i\sqrt{-k}$

Inclusion Reports:
The inclusion ratios of different sets of numbers are as follows:

$$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{D} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$$

Definitions :

Let a complex $z = a + ib$ (a et b réels)
- a is called the real part of z. It is also noted $Re(z)$.
- *b is called* ***the imaginary part of z****. It is also noted***.** $Im(z)$

We can therefore write **z** in the form: $z = Re(z) + iIm(z)$

A complex number has a unique real part and a **unique imaginary part.**

Ex :$z = 3 - i$ $Re(z) = 3$ et $Im(z) = -1$

A real number z is a complex number such that $\mathrm{Im}(z) = 0$ (imaginary part zero). A complex number z such $Re(z) = 0$ (zero real part) is said to be **pure imaginary.**

Properties :

$z = 0$ if and only if $a = 0$ **et** $b = 0$ (real and imaginary parts zero)

$a + ib = a' + ib'$ if and only if $a = a'$ **et** $b = b'$ (identical real parts and imaginary parts identical)

Operations :

$(a + ib) + (a' + ib') = (a + a') + i(b + b')$

$kz = ka + ikb$ for all real k

$zz' = (aa' - bb') + i(ab' + a'b)$

$\frac{1}{z} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$ pour $z \neq 0$

Exemples :

$(4 + 3i) + (7 - i) = 11 + 2i$
$(4 + 3i) \times (7 - i) = (28 + 3) + i(-4 + 21) = 31 + 17i$
$2(4 + 3i) = 8 + 6i$
$\frac{1}{4 + 3i} = \frac{4 - 3i}{16 + 9} = \frac{4}{27} - \frac{3}{27}i = \frac{4}{27} - \frac{1}{9}i$

Conjugated number:

The conjugate number of a complex $z = a + ib$ is the complex number equal to $a - ib$.
We note it. $\bar{z}$

Ex : $z = 6 + 2i$ $\bar{z} = 6 - 2i$

Conjugation properties:

$\overline{(\bar{z})} = z$

$\overline{z + z'} = \bar{z} + \bar{z'}$

$\overline{-z} = -\bar{z}$

$\overline{zz'} = \bar{z} \times \bar{z'}$

$\overline{z^n} = \bar{z}^n$ for all $n \in \mathbb{N}$

$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$ et $\overline{\left(\frac{z}{z'}\right)} = \frac{\bar{z}}{\bar{z'}}$ for $z \neq 0$

$z + \bar{z} = 2Re(z)$ and $z - \bar{z} = 2iIm(z)$ or $Re(z) = \frac{z+\bar{z}}{2}$ and $Im(z) = \frac{z-\bar{z}}{2i}$

2. Second degree polynomials

Let P be a second degree polynomial in C (i.e. defined for all $z \in \mathbb{C}$) **with real coefficients (real a, b and c):** $P(z) = az^2 + bz + c$

$\Delta = b^2 - 4ac$

-Si $\Delta > 0$ then the polynomial has two real roots: $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$

-Si $\Delta = 0$ then the polynomial has a double root: $\alpha = \frac{-b}{2a}$

- In R if $\Delta < 0$ then the polynomial has no root.
- In C if $\Delta < 0$ then the polynomial has two complex roots: $z_1 = \frac{-b-i\sqrt{-\Delta}}{2a}$ and $z_2 = \frac{-b+i\sqrt{-\Delta}}{2a}$

In C a second degree polynomial with real coefficients is therefore always factorable:
$P(x) = a(z - z_1)(z - z_2)$

Exemple :

$P(z) = 2z^2 + 2z + 1$

$\Delta = 2^2 - 4 \times 2 \times 1 = -4$ $\Delta < 0$ therefore P has two complex roots.

$z_1 = \frac{-2 - i\sqrt{4}}{2 \times 2}$ $z_2 = \frac{-2 + i\sqrt{4}}{2 \times 2}$

$z_1 = \frac{-2 - 2i}{4}$ $z_2 = \frac{-2 + 2i}{4}$

$z_1 = \frac{-1 - i}{2}$ $z_2 = \frac{-1 + i}{2}$

$P(z) = 2\left(z - \frac{-1 - i}{2}\right)\left(z - \frac{-1 + i}{2}\right)$

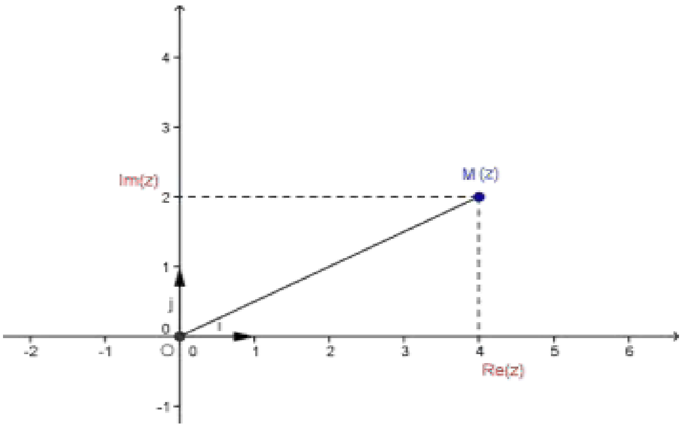
3. Module and argument

Graphic Representation :

A complex number z can be represented by a point M in a plane provided with a direct orthonormal coordinate system $(O, \vec{i}, \vec{j})$. Point M is the image of z in the plane.

Its abscissa corresponds to its real part and its ordinate to its imaginary part.

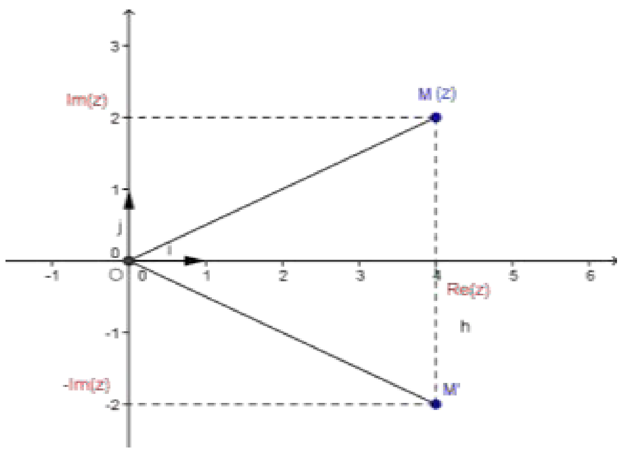
Ex : $z = 4 + 2i$



Noticed:

- The image of the conjugate $\bar{z}$ of a z complex is the symmetry of the point M with respect to the abscissa axis.

Ex : $z = 4 + 2i$
$\bar{z} = 4 - 2i$



Module:

The module of a complex $z = a + ib$ is the real number equal t$\sqrt{a^2 + b^2}$.
We note it $|z|$.

Geometric interpretation:

If a complex number z has as its image a point M then $|z| = OM$.
The module corresponds to the distance between the point M and the origin of the mark.
If a complex z has as its image a point M and a complex z' a point N then $|z - z'| = MN$ (distance between M and N)

Properties :

$|z| = |\bar{z}| = |-z| = |-\bar{z}|$

$z\bar{z} = |z|^2 = a^2 + b^2$ ($z\bar{z}$ is therefore a real number)

$|zz'| = |z| \times |z'|$

$|z^n| = |z|^n$ for evry $n \in \mathbb{N}$

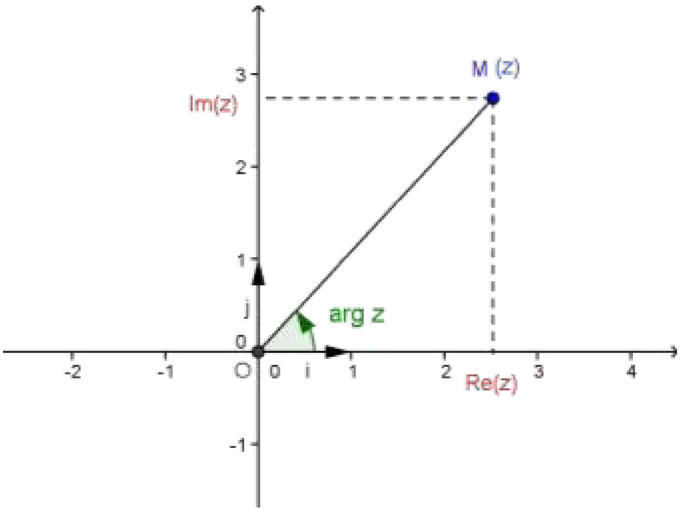
$\left|\frac{1}{z}\right| = \frac{1}{|z|}$ for $z \neq 0$

$\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|}$ For $z \neq 0$

$|z + z'| \leq |z| + |z'|$ (triangular inequality)

Argument :

An argument of a complex z having as image a point M is a value (in radians) of the angle $(\vec{i}, \overrightarrow{OM})$. **We note it $\arg z$.**



Remarks :

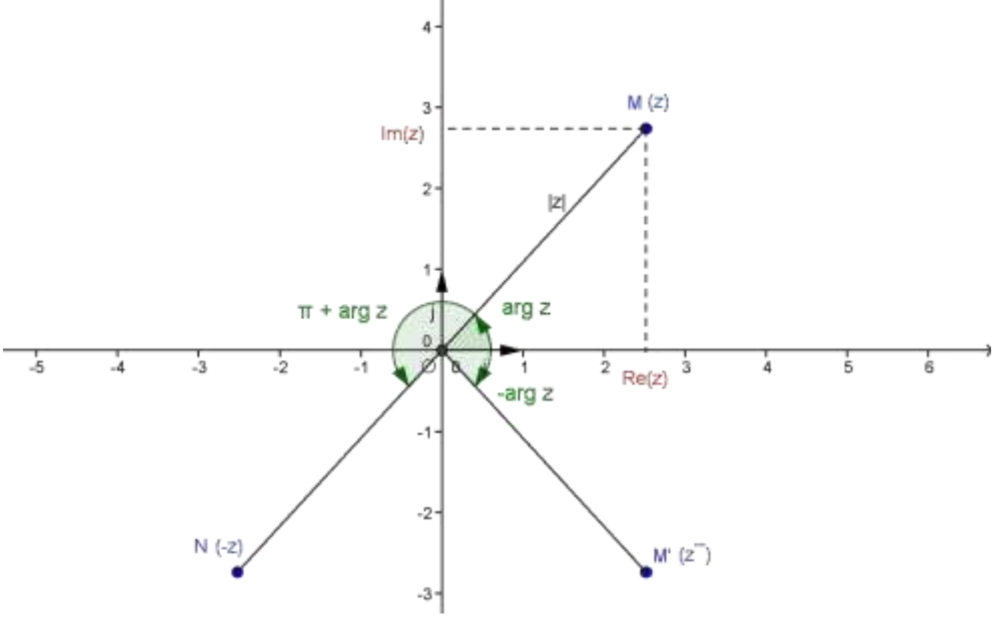
- - The number 0 has no argument.

- complex has an infinity of arguments: if is θ is an argument of z, then **$\theta + 2k\pi$ ($k \in \mathbb{Z}$) is also an argument of z.**

Properties :

$\arg(\bar{z}) = -\arg z + 2k\pi$ ($k \in \mathbb{Z}$)

$\arg(-z) = \pi + \arg z + 2k\pi$ ($k \in \mathbb{Z}$)



Trigonometric form:

consider θ an argument of z ($z \neq 0$).he abscissa of M corresponds to $|z|\cos\theta$ nd its ordinate to . $|z|\sin\theta$.We can therefore write z in the following trigonometric form: $\boldsymbol{z = |z|(\cos\theta + \sin\theta)}$

$\boldsymbol{Re(z) = |z|\cos\theta}$
$\boldsymbol{Im(z) = |z|\sin\theta}$

Noticed :

The real and imaginary parts are very unique because $\cos(\theta + 2k\pi) = \cos\theta$ and $\sin(\theta + 2k\pi) = \sin\theta$.

Exemple :

$z = 1 + i$
$|z| = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2}$
$|z| = \sqrt{2}$
$\cos\theta = \frac{a}{|z|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and $\sin\theta = \frac{b}{|z|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\theta = \frac{\pi}{4} + 2k\pi$

Property :

If two complex numbers are equal then they have the same module and same argumentc $2k\pi$ close

Operations:

$\boldsymbol{\arg\left(\frac{1}{z}\right) = -\arg z}$ for $z \neq 0$

$\boldsymbol{\arg(zz') = \arg z + \arg z'}$ $z \neq 0$

$\boldsymbol{\arg\left(\frac{z}{z'}\right) = \arg z - \arg z'}$ for $n \in \mathrm{N}$

$\boldsymbol{\arg(z^n) = n\arg z}$ for all

4.Exponential notation

The exponential notation of a complex $z = |z|(\cos\theta + \sin\theta)$ is $\boldsymbol{z = |z|e^{i\theta}}$.

Ex :

$z = -3 + 3i = 3\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 3\sqrt{2}e^{i\frac{3\pi}{4}}$

Remarks :

- A complex number with module 1 is of the form $e^{i\theta}$.
- we have : $\boldsymbol{e^{i\pi} = -1}$
- The number 0 does not have exponential notation (it has no argument).

Property :

if $r_1 e^{i\theta_1} = r_2 e^{i\theta_2}$ with $r_1 \in$ R et $r_2 \in$ R then $r_1 = r_2$ et $\theta_1 = \theta_2 + 2k\pi$.

Conjugated:

The conjugate of a complex number $z = |z|e^{i\theta}$ is $z = |z|e^{-i\theta}$.

Operations :

$zz' = |z||z'|e^{i(\theta+\theta')}$

$\frac{1}{z} = \frac{1}{|z|}e^{-i\theta}$ for $z \neq 0$

$\frac{z}{z'} = \frac{|z|}{|z'|}e^{i(\theta-\theta')}$ for $z \neq 0$

$z^n = |z|^n e^{in\theta}$ for all $n \in$ N