Model Solution & Marking Scheme

Final Exam – Analysis 2

Total Marks: 20
Duration: 120 minutes

Important Note: The marking scheme indicates the breakdown of points for each step. Full marks are awarded for correct final answers with proper justification, regardless of the valid method used.

Exercise 1: Differential Equation Classification & IVP (6 points)

Question 1: Classification of $x^2y'' + xy' + (x^2 - n^2)y = 0$ (2 points)

- **Order:** 2 (0.5 pt)– highest derivative is y''.
- **Dependent variable:** y (0.5 pt).
- **Independent variable:** x (0.5 pt).
- **Type:** Ordinary Differential Equation (ODE) (0.5 pt) – no partial derivatives.

Question 2: Solution of IVP $\dfrac{dy}{dx} + 3y = 8,\ y(0) = 2$ (4 points)

Method 1 (Separation of variables):

- Rewrite as $\dfrac{dy}{dx} = 8 - 3y$. (0.5 pt)
- Separate variables: $\dfrac{dy}{8-3y} = dx$. (0.5 pt)
- Integrate both sides: $-\dfrac{1}{3}\ln|8-3y| = x + C$. (1 pt)
- Simplify: $\ln|8-3y| = -3x - 3C \Rightarrow 8 - 3y = Ae^{-3x}$ where $A = \pm e^{-3C}$. (0.5 pt)
- Solve for y: $y = \frac{8}{3} - \frac{A}{3}e^{-3x}$. Or directly $y = \frac{8}{3} + Ce^{-3x}$ after redefining constant. (0.5 pt)
- Apply $y(0) = 2$: $2 = \frac{8}{3} + C \Rightarrow C = -\frac{2}{3}$. (0.5 pt)
- Final: $\boxed{y = \dfrac{8}{3} - \dfrac{2}{3}e^{-3x}}$. (0.5 pt)

Method 2 (Integrating factor):

- Write in standard linear form: $y' + 3y = 8$. (0.5 pt)

- Compute integrating factor: $\mu(x) = e^{\int 3\,dx} = e^{3x}$. (0.5 pt)
- Multiply ODE by μ and rewrite as $\frac{d}{dx}\left(e^{3x}y\right) = 8e^{3x}$. (0.5 pt)
- Integrate both sides: $e^{3x}y = \frac{8}{3}e^{3x} + C$. (1 pt)
- Solve for y: $y = \frac{8}{3} + Ce^{-3x}$. (0.5 pt)
- Use initial condition $y(0) = 2$: $2 = \frac{8}{3} + C \Rightarrow C = -\frac{2}{3}$. (0.5 pt)
- Final solution: $\boxed{y = \frac{8}{3} - \frac{2}{3}e^{-3x}}$. (0.5 pt)

Note: Either method is acceptable. Marks are given for proper separation/integration, handling the constant, and applying the initial condition.

Exercise 2: Bernoulli Equation & Second-Order Linear ODE (8 points)

Question 1: Bernoulli equation $y' + xy = x^3y^3$ (5 points)

- Recognize $n = 3$, divide by y^3: $y^{-3}y' + xy^{-2} = x^3$. (0.5 pt)
- Substitute $z = y^{-2} \Rightarrow z' = -2y^{-3}y' \Rightarrow y^{-3}y' = -\frac{1}{2}z'$.(0.5 pt)
- Linear ODE: $z' - 2xz = -2x^3$. (0.5 pt)
- Integrating factor $\mu = e^{\int -2x dx} = e^{-x^2}$. (0.5 pt)
- $\frac{d}{dx}(ze^{-x^2}) = -2x^3e^{-x^2}$. (0.5 pt)
- Integrate RHS using $u = x^2$: $\int -2x^3e^{-x^2}dx = (x^2+1)e^{-x^2} + C$. (1 pt)
- Thus $ze^{-x^2} = (x^2+1)e^{-x^2} + C \Rightarrow z = x^2 + 1 + Ce^{x^2}$. (0.5 pt)
- Back-substitute: $y^{-2} = x^2 + 1 + Ce^{x^2}$. (0.5 pt)
- Final: $\boxed{y^2 = \frac{1}{x^2 + 1 + Ce^{x^2}}}$ (or equivalently $y^{-2} = x^2 + 1 + Ce^{x^2}$). (0.5 pt)

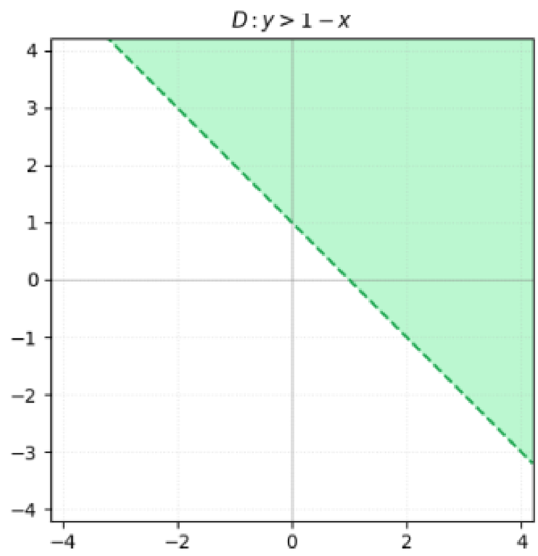
Question 2: General solution of $y'' - 3y' + 2y = 0$ (3 points)

- Write the characteristic equation: $r^2 - 3r + 2 = 0$. (0.5 pt)
- Solve for the roots: $(r-1)(r-2) = 0 \Rightarrow r_1 = 1,\ r_2 = 2$ (real and distinct). (1.5 pt)
- Form the general solution for distinct real roots and present the final answer: $\boxed{y = C_1e^x + C_2e^{2x}}$. (1 pt)

Exercise 3: Domain, Partial Derivatives, Double Integral (6 points)

Question 1: Domain of $f(x, y) = \ln(x+y-1)$ and sketch (2 points)

- Condition: $x + y - 1 > 0 \Rightarrow y > 1 - x$. (0.5 pt)
- Domain: open half-plane above the line $y = -x + 1$ (line excluded). (0.5 pt)
- Sketch: Draw dashed line through $(0, 1)$ and $(1, 0)$, shade above. (1 pt)



Question 2: $f(x, y) = 3x^2 + 4xy - 2y^2$ at $(2, -3)$ and partial derivatives (2.5 points)

(a) Function value: $f(2, -3) = 3(2)^2 + 4(2)(-3) - 2(-3)^2 = 12 - 24 - 18 = -30$. (0.5 pt)

(b) $f_x(2, -3)$: $f_x = 6x + 4y \Rightarrow 6(2) + 4(-3) = 0$. (0.5 pt)

(c) $f_y(2, -3)$: $f_y = 4x - 4y \Rightarrow 4(2) - 4(-3) = 20$. (0.5 pt)

(d) $f_{xx}(2, -3)$: $f_{xx} = 6$ (constant) $\Longrightarrow 6$. (0.25 pt)

(e) $f_{xy}(2, -3)$: $f_{xy} = 4$ (constant) $\Longrightarrow 4$. (0.25 pt)

(f) $f_{yx}(2, -3)$: by Clairaut's theorem equals f_{xy}, so 4. (0.25 pt)

(g) $f_{yy}(2, -3) = -4$. (0.25 pt)

Point distribution: (a) 0.5 pt. First-order partials (b) and (c): 0.5 pt each because they require differentiation + substitution. Second-order partials (d)–(g): 0.25 pt each as they are simply constant values (second derivatives of a quadratic). Total = 0.5 + 1.0 + 1.0 = 2.5 pts.

Question 3: Double integral $\iint_R (3x + 2y)\, dA$ over $R = [1, 3] \times [0, 2]$ (1.5 points)

- Set up iterated integral: $\int_1^3 \int_0^2 (3x + 2y)\, dy\, dx$. (0.5 pt)
- Inner: $\int_0^2 (3x + 2y) dy = [3xy + y^2]_0^2 = 6x + 4$. (0.5 pt)
- Outer: $\int_1^3 (6x + 4) dx = [3x^2 + 4x]_1^3 = (27 + 12) - (3 + 4) = 32$. (0.5 pt)
- Final: $\boxed{32}$.

End of Correction