Geometric Characteristics of Sections

Properties of Areas

In this chapter, we will focus on the following characteristics:

- Area of a section
- Center of gravity
- Static moment
- moment of inertia (quadratic moment)

Area of a Section

By definition, the area A of a section is given by the integral: $A = \int_A dA$

A ***centroid*** is the geometric center of a geometric object: a one-dimensional curve, a two-dimensional area or a three-dimensional volume. Centroids are useful for many situations in Statics and subsequent courses, including the analysis of distributed forces, beam bending, and shaft torsion.

Two related concepts are the ***center of gravity***, which is the average location of an object's *weight,* and the ***center of mass*** which is the average location of an object's *mass*. In many engineering situations, the centroid, center of mass, and center of gravity are all coincident. Because of this, these three terms are often used interchangeably without regard to their precise meanings.

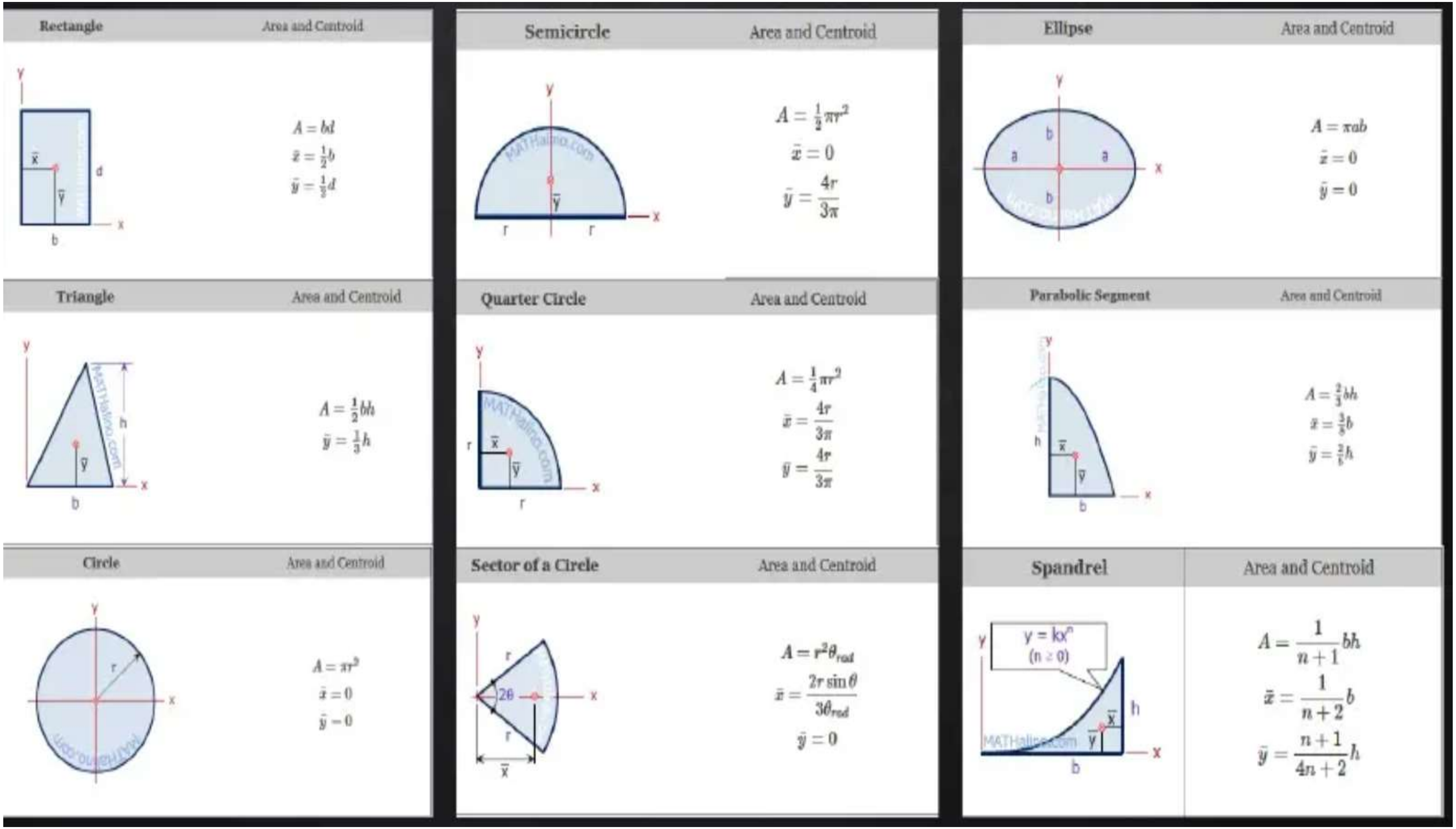
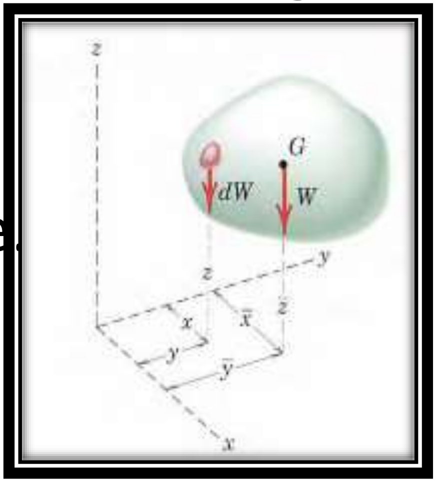

Rectangle
Area and Centroid
$A = bd$
$\bar{x} = \frac{1}{2}b$
$\bar{y} = \frac{1}{2}d$
Semicircle
Area and Centroid
$A = \frac{1}{2}\pi r^2$
$\bar{x} = 0$
$\bar{y} = \frac{4r}{3\pi}$
Ellipse
Area and Centroid
$A = \pi ab$
$\bar{x} = 0$
$\bar{y} = 0$
Triangle
Area and Centroid
$A = \frac{1}{2}bh$
$\bar{y} = \frac{1}{3}h$
Quarter Circle
Area and Centroid
$A = \frac{1}{4}\pi r^2$
$\bar{x} = \frac{4r}{3\pi}$
$\bar{y} = \frac{4r}{3\pi}$
Parabolic Segment
Area and Centroid
$A = \frac{2}{3}bh$
$\bar{x} = \frac{3}{8}b$
$\bar{y} = \frac{2}{5}h$
Circle
Area and Centroid
$A = \pi r^2$
$\bar{x} = 0$
$\bar{y} = 0$
Sector of a Circle
Area and Centroid
$A = r^2\theta_{rad}$
$\bar{x} = \frac{2r\sin\theta}{3\theta_{rad}}$
$\bar{y} = 0$
Spandrel
Area and Centroid
$y = kx^n$
$(n \geq 0)$
$A = \frac{1}{n+1}bh$
$\bar{x} = \frac{1}{n+2}b$
$\bar{y} = \frac{n+1}{4n+2}h$

Figure.1

Center of Gravity

- Two related concepts are the ***center of gravity***, which is the average location of an object's *weight,* and the ***center of mass*** which is the average location of an object's *mass*. In many engineering situations, the centroid, center of mass, and center of gravity are all coincident. Because of this, these three terms are often used interchangeably without regard to their precise meanings.



- A body is composed of an infinite number of particles of differential size.
- If these particles have a weight *dW*, they will approximately
- form a parallel force system, and the resultant of this system is the total weight of the body *W.* This weight will pass through a single point which is the center of gravity *G*.

Example .1 Calculate the area of a triangle.

Solution .1 Consider the planar triangular surface shown in the figure.1.

We have an elementary surface such that:

$$dA = h\left(1 - \frac{x}{b}\right)dx$$

$$A = \int_A dA = \int_0^b h\left(1 - \frac{x}{b}\right)dx = \frac{bh}{2}$$

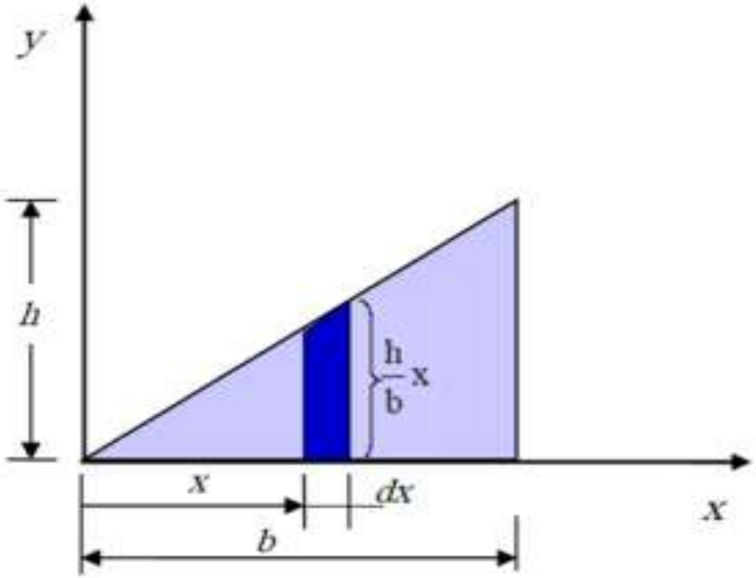


Figure.2

Static Moment

The static moment S of a section respect to an axis ox or *oy* (Fig. 3.) is given by one of the following expressions.

$$S_x = \int y\,dA = \bar{y}A \text{ and } S_y = \int x\,dA = \bar{x}A,$$

If translations are made parallel to the *ox* and *oy* axes, the static moments change. Let the section shown in figure 4. be such that S_X, S_Y, and A are known, and we aim to determine $S_{X'}$ and $S_{Y'}$.

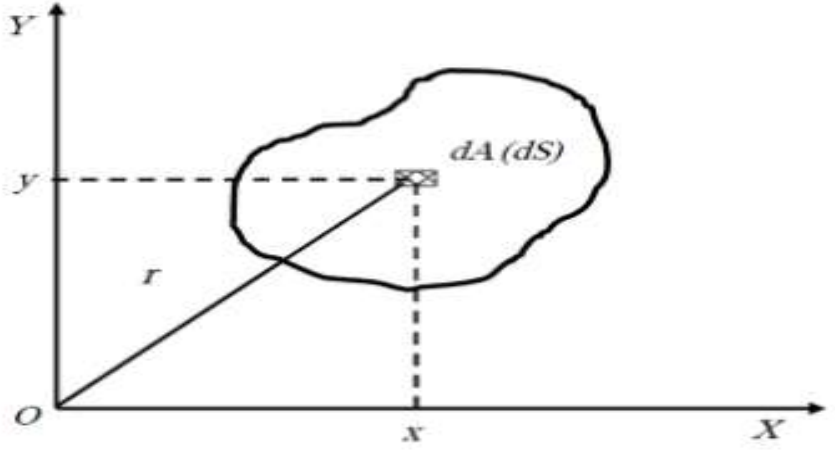


Figure.3

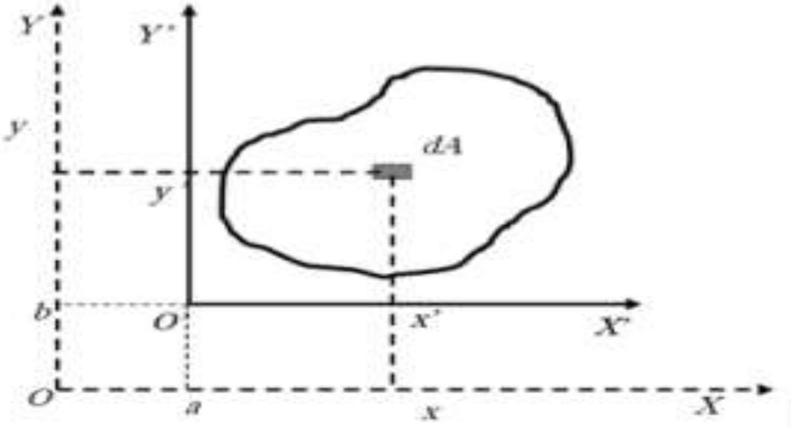


Figure.4

From figure .3, we have: By definition, we have: Where the center of gravity $x = x_a$, $y = x_b$ by definition

$$S_{x'} = \int y dA = \int (y - b) dA$$

Where

$$S_{X'} = S_x - b.A,\ S_{Y'} = S_y - a.A$$

Center of Gravity

The moment of the resultant force *W* about any axis equals to the sum of the moments of the weights *dw* about the same axis.

$$\text{Equating the moments about the } y\text{-axis}: \bar{x}W = \Sigma x dw = \int x dw$$

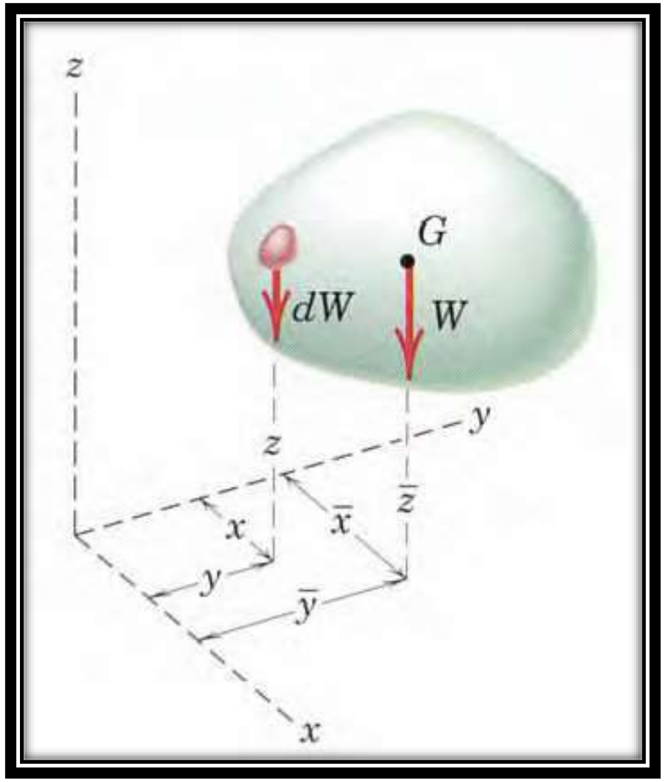
$$\text{Equating the moments about the } x\text{-axis}: \bar{y}W = \Sigma y dw = \int y dw$$

$$\text{Rotating the axes by } 90^\circ \text{ about the } y\text{-axis and then}$$

$$\text{equating the moments about the } y\text{-axis}: \bar{z}W = \Sigma z dw = \int z dw$$

$\therefore$ The C.G. with respect to the $x, y,$ and z axes is

$$\boxed{\bar{x} = \frac{\int x dw}{W}; \quad \bar{y} = \frac{\int y dw}{W}; \quad \bar{z} = \frac{\int z dw}{W}}$$



Center of Gravity versus Centroid

Centroid is the point, where whole area of the plane is going to be act. (mostly applicable for two dimensional problems)
Center of gravity is the point, where whole weight of the body is going to be act. (mostly applicable for three dimensional problems)

There are two major differences between "**center of gravity**" and "**centroid**":

1) The term "center of gravity" applies to the bodies with mass and weight, while the term "centroid" applies to plane areas.

2) Center of gravity of a body is the point through which the resultant gravitational force (weight) of the body acts for any orientation of the body, while centroid is the point in a plane area such that the moment of the area, about any axis, through that point is zero.

Center of Mass

If gravity field is treated as uniform and parallel, *g* is constant.

$$W = mg; dw = (dm)g$$

The expression for C.G. modifies to :

$$\bar{x} = \frac{g\int x\,dm}{gm}; \quad \bar{y} = \frac{g\int y\,dm}{gm}; \quad \bar{z} = \frac{g\int z\,dm}{gm}$$

$$\Rightarrow \boxed{\bar{x} = \frac{\int x\,dm}{m}; \quad \bar{y} = \frac{\int y\,dm}{m}; \quad \bar{z} = \frac{\int z\,dm}{m}}$$

This expression contain no reference to gravitational effects since *g* no longer appears. Therefore, it defines a unique point in the body which is a function solely of the distribution of mass. This point is known as the CENTER OF MASS, and clearly it coincides with the center of gravity as long as the gravity field is uniform and parallel.

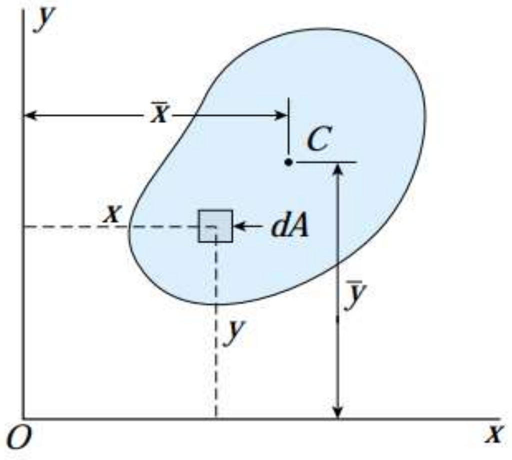
It is meaningless to speak of the center of gravity of a body that is removed from the earth's gravitational field, since no gravitational forces would act on the body. The body would, however, still possess its unique center of mass.

If the area is doubly symmetric about two orthogonal axes, the centroid lies at the intersection of those axes. If the area is symmetric about only one axis, then the centroid lies somewhere along that axis (the other coordinate will need to be calculated).

If the exact location of the centroid cannot be determined by inspection, it can be calculated by:

Centroids of Plane Areas

Fig. shows a plane area of irregular shape with its centroid at point ***C***. The ***xy*** coordinate system is oriented arbitrarily with its origin at any point ***O***.



The **area** of the geometric figure is defined by the following integral:

In which ***dA*** is a differenti nt of area having coordinates ***x*** and ***y***

$$A = \int dA$$

The **first moments** of the area with axes are defined, respectively, as follows:

$$Q_x = \int y\,dA \qquad Q_y = \int x\,dA$$

The coordinates *x* w and *y* w of the **centroid** *C* (I the first moments divided by the area:

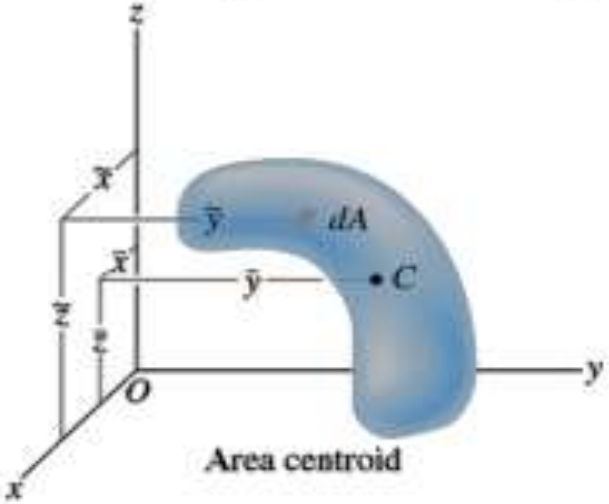
$$\bar{x} = \frac{Q_y}{A} = \frac{\int x\,dA}{\int dA} \qquad \bar{y} = \frac{Q_x}{A} = \frac{\int y\,dA}{\int dA}$$

Centroids of an Area

Centroid or Centre of area

The point at which area of the plane figure (such as rectangle, square, circle…..) is assumed to be concentrated is called the centroid of that area . The centroid is also represented by C.G or G.

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} \qquad \bar{z} = \frac{\int_A \tilde{z}\, dA}{\int_A dA}$$



STEPS TO DETERMINE THE CENTROID OF AN AREA

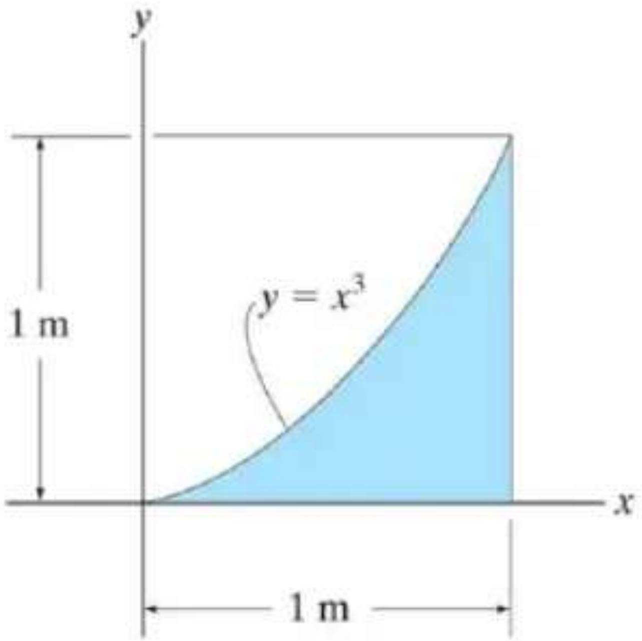
1. Choose an appropriate differential element dA at a general point (x,y). Hint: Generally, if y is easily expressed in terms of x (e.g., $y = x^2 + 1$), use a vertical rectangular element. If the converse is true, then use a horizontal rectangular element.

2. Express dA in terms of the differentiating element dx (or dy).

3. Determine coordinates $(\tilde{x}, \tilde{y})$ of the centroid of the rectangular element in terms of the general point (x,y).

4. Express all the variables and integral limits in the formula using either x or y depending on whether the differential element is in terms of dx or dy, respectively, and integrate.

Example 2



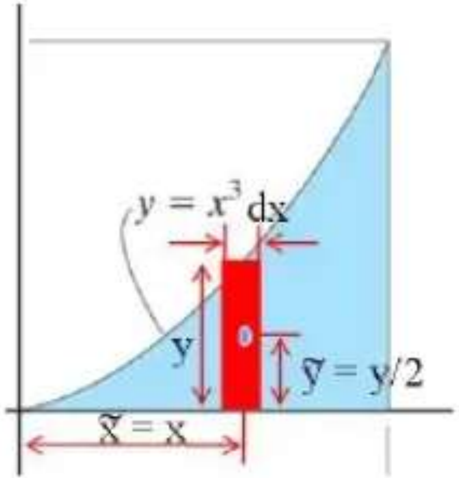
Given: The area as shown.

Find: The centroid location $(\bar{x}, \bar{y})$

Plan: Follow the steps.

Solution:

1. Since y is given in terms of x, choose dA as a vertical rectangular strip.



2. $dA = y\,dx = x^3\,dx$

3. $\tilde{x} = x$ and $\tilde{y} = y/2 = x^3/2$

4. $\bar{x} = \left(\int_A \tilde{x} \, dA \right) / \left(\int_A dA \right)$

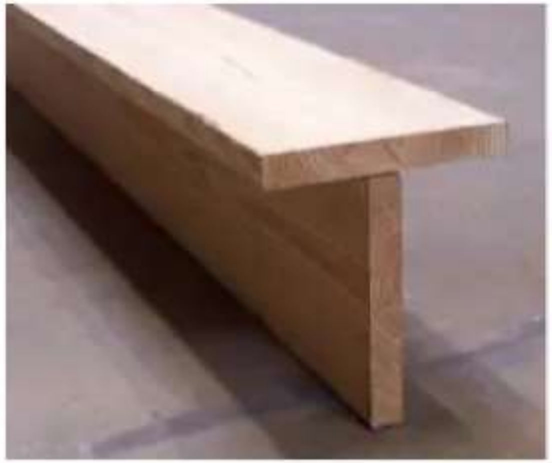
$$= \frac{{}_0\int^1 x\,(x^3)\,d\,x}{{}_0\int^1 (x^3)\,d\,x} = \frac{1/5\,[\,x^5\,]_0^1}{1/4\,[\,x^4\,]_0^1}$$

$= (1/5) / (1/4) = 0.8\ \text{m}$

$$\bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} = \frac{{}_0\int^1 (x^3/2)\,(x^3)\,dx}{{}_0\int^1 x^3\,dx} = \frac{1/14[x^7]_0^1}{1/4}$$

$= (1/14) / (1/4) = 0.2857\ \text{m}$

APPLICATIONS



The I-beam (top) or T-beam (bottom) shown are commonly used in building various types of structures.

When doing a stress or deflection analysis for a beam, the location of its centroid is very important.

How can we easily determine the location of the centroid for different beam shapes?

STEPS FOR ANALYSIS

1. Divide the body into pieces that are known shapes.
   Holes are considered as pieces with negative weight or size.

2. Make a table with the first column for segment number, the second column for size, the next set of columns for the moment arms, and, finally, several columns for recording results of simple intermediate calculations.

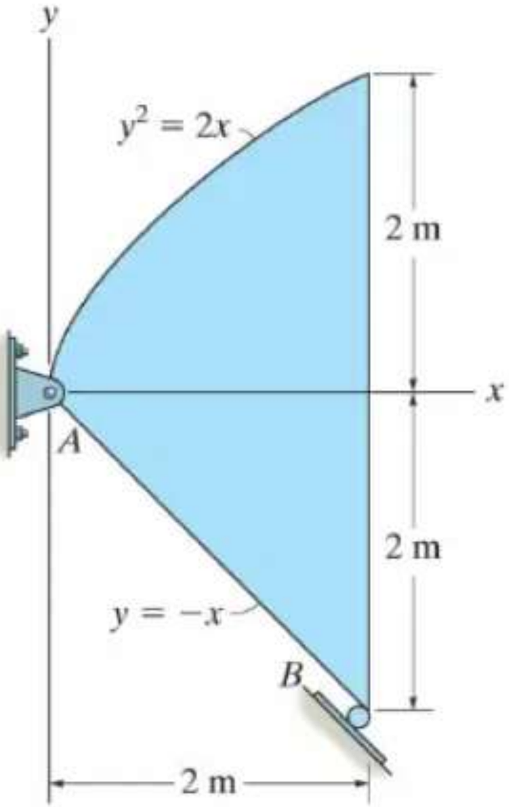
3. Fix the coordinate axes, determine the coordinates of centroid of each piece, and then fill in the table.

4. Sum the columns to get $\bar{x}$, $\bar{y}$, and $\bar{z}$. Use formulas like

   $$\bar{x} = (\Sigma \tilde{x}_i A_i) / (\Sigma A_i)$$

   This approach will become straightforward by doing examples!

GROUP PROBLEM SOLVING



Given: A plate as shown.

Find: The location of its centroid

Plan:

Follow the solution steps to find the centroid by integration.
(1) Define *dA* in the blue area;
(2) Find *x*- and *y*-centroid in *dA*;
(3) Integrate *dA* in the domain to find the total area;
(4) Use integration to find *x*- and *y*-centroids for the entire shape (left equations).

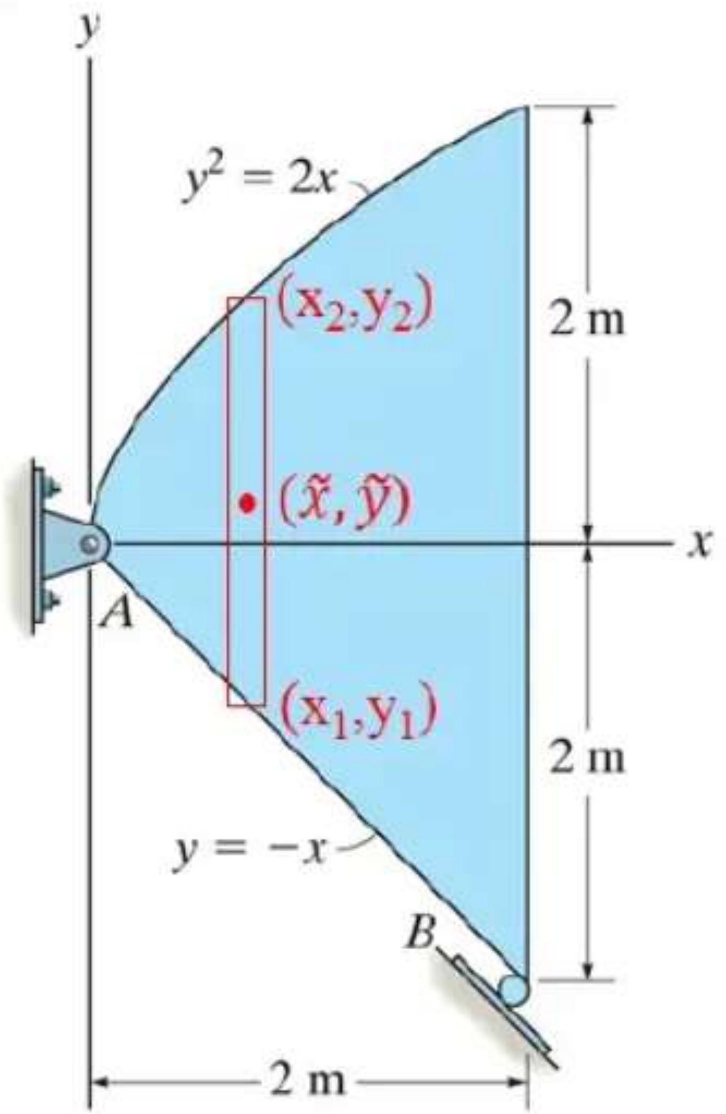
$$\bar{x} = \frac{\int \tilde{x}\, dA}{\int dA} \qquad \bar{y} = \frac{\int \tilde{y}\, dA}{\int dA}$$

Solution

1. Choose dA as a vertical rectangular strip.

2. $dA = (y_2 - y_1)\ dx$
$$= (\sqrt{2x} + x)\ dx$$

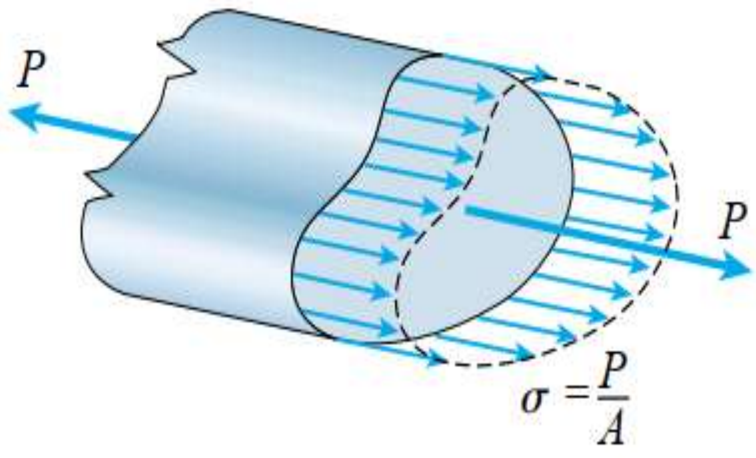
3. $\tilde{x} = x$
$$\tilde{y} = (y_1 + y_2) / 2$$
$$= (\sqrt{2x} - x) / 2$$



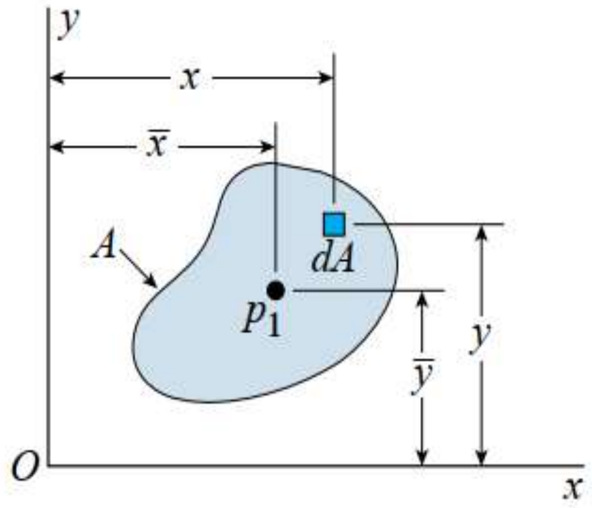
Line of Action of the Axial Forces for a Uniform Stress Distribution

In a prismatic bar, we assumed that the normal stress σ was distributed uniformly over the cross section. Now we will demonstrate that this condition is met if the line of action of the axial forces is through the centroid of the cross-sectional area.

Consider a prismatic bar of arbitrary cross-sectional shape subjected to axial forces $\boldsymbol{P}$ that produce uniformly distributed stresses σ (Fig.).

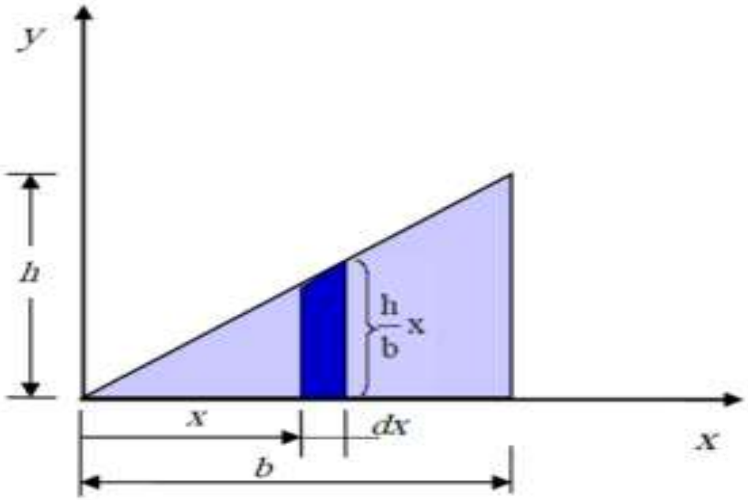


(a)



(b)

Example 3 Determine the coordinates of the cent of gravity of the triangular section shown below.



Solution

$$X_G = \frac{\int_A xdA}{\int_A dA} = \frac{\int_0^b x\left(\frac{h}{b}xdx\right)}{\int_0^b \frac{h}{b}xdx}$$

Where:

$$X_G = \frac{2}{3}b$$

$$Y_G = \frac{\int_A ydA}{\int_A dA} = \frac{\int_0^b x\left(\frac{h}{b}xdx\right)}{\int_0^b \frac{h}{b}xdx}$$

Where:

$$Y_G = \frac{1}{3}h$$

The moments of the force *P* are $M_x = P\bar{y} \qquad M_y = -P\bar{x}$

In which a moment is considered positive when its vector (using the right-hand rule) acts in the positive direction of the corresponding axis.*

The moments of the distributed stresses are obtained by integrating over the cross-sectional area ***A*** .

- The differential force acting on an element of area *dA* (Fig. 1-4b) is equal to σ ***dA***.
- The moments of this elemental force about the *x* and *y* axes are σ ***ydA*** and $-\sigma$ ***xdA***, respectively, in which ***x*** and ***y*** denote the coordinates of the element *dA.*
- The total moments are obtained by integrating over the cross-sectional area:

$$M_x = \int \sigma y\, dA \qquad M_y = -\int \sigma x\, dA$$

These expressions give the moments produced by the stresses s. Next, we equate the moments ***Mx*** and ***My*** as obtained from the force ***P*** (Eqs. a and b) to the moments obtained from the distributed stresses (Eqs. C and d):

$$P\bar{y} = \int \sigma y\, dA \qquad P\bar{x} = \int \sigma x\, dA$$

Because the stresses σ are uniformly distributed, we know that they are constant over the cross-sectional area *A* and can be placed outside the integral signs.

Also, we know that σ is equal to *P/A*.

Therefore, we obtain the following formulas for the coordinates of point *p1*:

$$\bar{y} = \frac{\int y \, dA}{A} \qquad \bar{x} = \frac{\int x \, dA}{A}$$

These equations are the same as the equations defining the coordinates of the centroid of an area

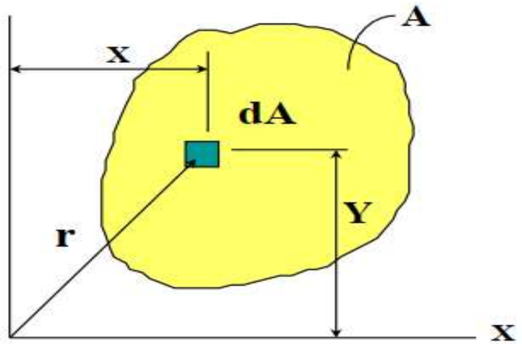
Also, let ***p1*** represent the point in the cross section where the line of action of the forces intersects the cross section (Fig. 2-1b). We construct a set of *xy* axes in the plane of the cross section and denote the coordinates of point *p1* by $\bar{y}$ and x. To determine these coordinates, we observe that the moments Mx and My of the force P about the x and y axes, respectively, must be equal to the corresponding moments of the uniformly distributed stresses.

- **Moment of Inertia (MoI) (second moment of an area (m4)**
- **Definition of Moments of Inertia for Areas**: Used in formulas for Mechanics of Materials, Fluid Mechanics, Structural Mechanics
- The moment of inertia is found by integrating

$$I_x = \int_{\text{Area}} y^2 dA$$

$$I_y = \int_{\text{Area}} x^2 dA$$

$$j_o = \int_{\text{Area}} \mathbf{r}^2 dA$$



Polar Moment of Inertia The moment of inertia of dA about the pole O or z axis is referred as the polar moment.it is defined as:- dJO = r2dA , where r is the perpendicular distance from the pole(z axis) to the element dA
For the entire area the polar moment of inertia is:-

$$J_O = \int_A r^2 \, dA = \int_A (x^2 + y^2) \, dA = I_x + I_y$$

J o is the polar moment of inertia about the pole O or Z axis

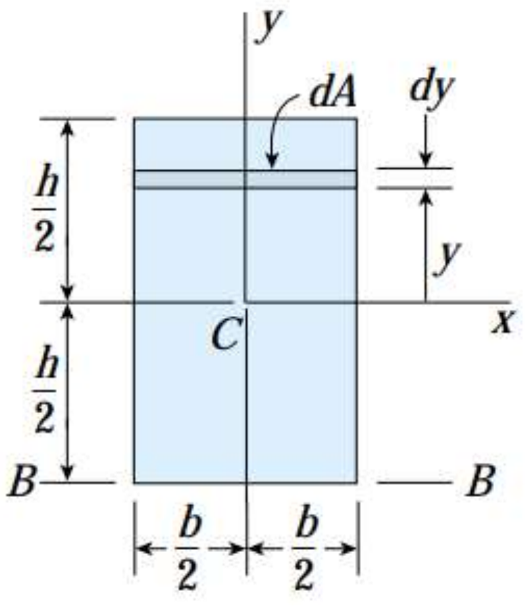
To illustrate how moments of inertia are obtained by integration, we will consider a rectangle having width *b* and height *h* (Fig. 12-10). The *x* and *y* axes have their origin at the centroid *C*. For convenience, we use a differential element of area *dA* in the form of a thin horizontal strip of width *b* and height *dy* (therefore, *dA* = *bdy*). Since all parts of the elemental strip are the same distance from the *x* axis, we can express the moment of inertia *Ix* with respect to the *x* axis as follows:

$$I_x = \int y^2 dA = \int_{-h/2}^{h/2} y^2 b\, dy = \frac{bh^3}{12} \qquad I_y = \int x^2 dA = \int_{-b/2}^{b/2} x^2 h\, dx = \frac{hb^3}{12}$$

In a similar manner, we can use an element of area in the form of a vertical strip with area *dA* *h dx* and obtain the moment of inertia with respect to the *y* axis:

If a different set of axes is selected, the moments of inertia will have different values. For instance, consider axis *BB* at the base of the rectangle (Fig. 12-10). If this axis is selected as the reference, we must define *y* as the coordinate distance from that axis to the element of area *dA*. Then the calculations for the moment of inertia become

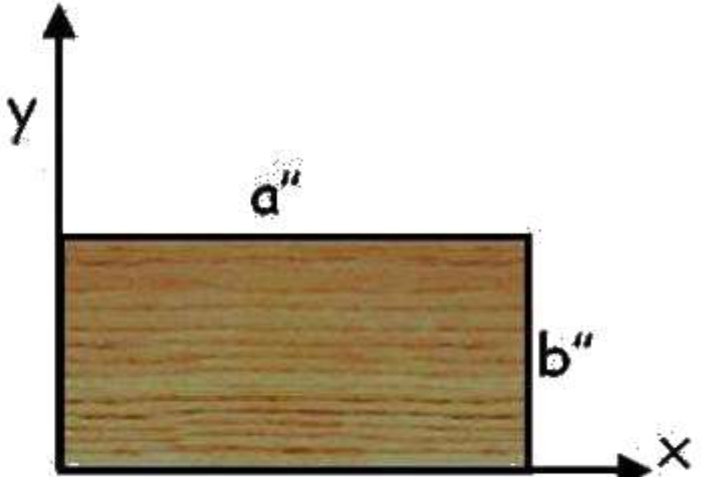
$$I_{BB} = \int y^2 dA = \int_0^h y^2 b\, dy = \frac{bh^3}{3}$$



EXAMPLE

Given: The shaded area shown in the figure.

Find: The MoI of the area about the x- and y-axes.



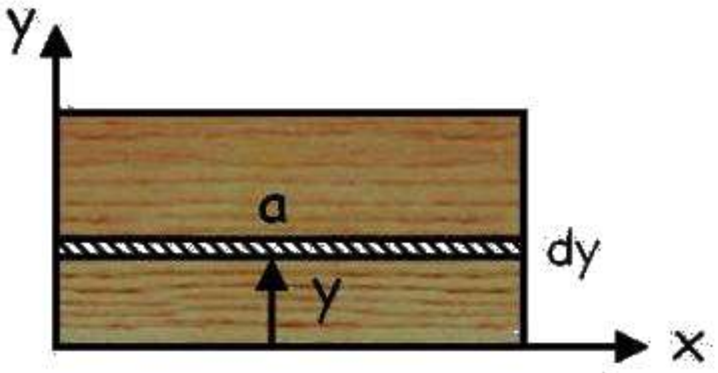
Solution

$I_x = \int y^2 \, dA$

$dA = a.\, dy$

$I_x = \int_0^b y^2 \,.\, a.\, dy$

$= [\, a \,.y^3/3]_0^b = ab^3/3 \;\; in^4$

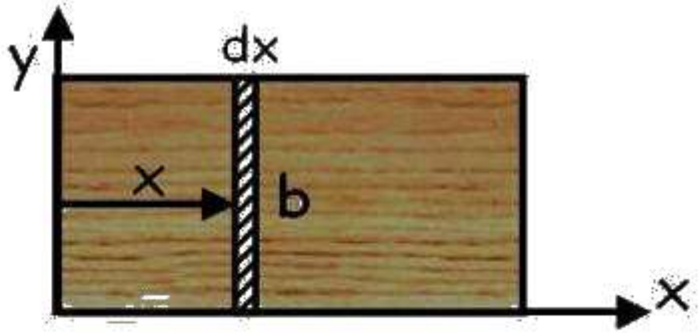


$I_y = \int x^2 \, dA$

$dA = b.\, dx$

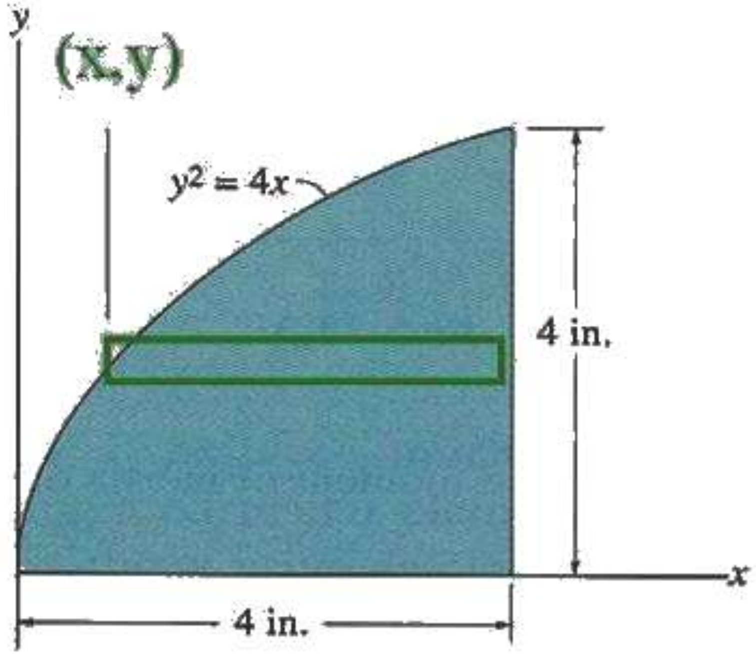
$I_y = \int_0^a x^2 \,.\, b.\, dx$

$= [\, b \,.x^3/3]_0^a = ba^3/3 \;\; in4$



Given: The shaded area shown in the figure.

Find: The MoI of the area about the x- and y-axes.



Solution

$$I_x = \int y^2 \, dA$$

$$dA = (4 - x)\, dy = (4 - y^2/4)\, dy$$

$$I_x = \int_0^4 y^2 \, (4 - y^2/4)\, dy$$

$$= \left[(4/3)\, y^3 - (1/20)\, y^5 \right]_0^4 = 34.1 \text{ in}^4$$