Analysis II: Solutions of Tutorial Exercise Sheet 3

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Exercise 01: Double Integrals and Change of Order

1. (a) $\iint_R (3x+2y)\,dA,\ R=[1,3]\times[0,2]$

$$\int_1^3\int_0^2 (3x+2y)\,dy\,dx = \int_1^3 \big[3xy+y^2\big]_0^2 dx = \int_1^3 (6x+4)\,dx = \big[3x^2+4x\big]_1^3 = 32.$$

   (b) $\iint_R y\sin x\,dA,\ R=[0,\pi]\times[0,1]$

$$\int_0^\pi\int_0^1 y\sin x\,dy\,dx = \int_0^\pi \sin x\left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{2}\int_0^\pi \sin x\,dx = \frac{1}{2}\big[-\cos x\big]_0^\pi = 1.$$

2. **Area enclosed by $y=x^2$ and $y=2x$:**
   Intersection: $x^2=2x \Rightarrow x(x-2)=0 \Rightarrow x=0,2$. For $x\in[0,2]$, $2x\geq x^2$.
   Area $= \iint_R dA = \int_{x=0}^2\int_{y=x^2}^{2x} dy\,dx = \int_0^2 (2x-x^2)\,dx = \left[x^2-\frac{x^3}{3}\right]_0^2 = 4-\frac{8}{3} = \frac{4}{3}.$

3. **Polar coordinates:** $\iint_R (x^2+y^2)\,dA,\ R: x^2+y^2\leq 9.$
   Transformation: $x=r\cos\theta,\ y=r\sin\theta$.
   Jacobian matrix:

$$= \begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}$$

   Jacobian determinant:

$$J=\cos\theta\cdot r\cos\theta-(-r\sin\theta)\cdot\sin\theta = r(\cos^2\theta+\sin^2\theta)=r.$$

   Thus $dA=dx\,dy=|J|\,dr\,d\theta=r\,dr\,d\theta$.
   Integration:

$$\iint_R (x^2+y^2)\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} r^2\cdot r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_0^3 r^3dr = 2\pi\cdot\frac{81}{4} = \frac{81\pi}{2}.$$

Exercise 02: Applications of Double Integrals

1. **(Center of gravity)** Semicircular plate $x^2+y^2\leq R^2,\ y\geq 0$, uniform density $\rho=1$.
   By symmetry $\bar{x}=0$. Area $A=\frac{\pi R^2}{2}$.

$$\bar{y}=\frac{1}{A}\iint_R y\,dA = \frac{2}{\pi R^2}\int_{x=-R}^{R}\int_{y=0}^{\sqrt{R^2-x^2}} y\,dy\,dx.$$

$$\int_0^{\sqrt{R^2-x^2}} y\,dy = \frac{R^2-x^2}{2},\qquad \int_{-R}^{R}\frac{R^2-x^2}{2}dx = \frac{1}{2}\left[2R^3-\frac{2R^3}{3}\right] = \frac{2R^3}{3}.$$

   Thus $\bar{y}=\frac{2}{\pi R^2}\cdot\frac{2R^3}{3}=\frac{4R}{3\pi}$. Center of gravity: $\left(0,\frac{4R}{3\pi}\right)$.

2. **(Moments of Inertia)** Rectangle $[0,3]\times[0,2]$, $\rho=1$.

$$I_y=\iint_R x^2\,dA=\int_0^3\int_0^2 x^2\,dy\,dx=\int_0^3 2x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^3 = 2\cdot\frac{27}{3}=18.$$

$$I_x=\iint_R y^2\,dA=\int_0^3\int_0^2 y^2\,dy\,dx=\int_0^3\left[\frac{y^3}{3}\right]_0^2 dx=\int_0^3\frac{8}{3}\,dx=\frac{8}{3}\cdot 3=8.$$

   Comparison: $I_y=18>I_x=8$, since rectangle is longer in x-direction.

3. **(Engineering - Total Mass)** Triangular plate vertices $(0,0),(4,0),(0,3)$, $\sigma(x,y)=x+y$.
Region: $x\geq 0$, $y\geq 0$, $y\leq 3-\frac{3}{4}x$ (since line from $(4,0)$ to $(0,3)$: $y=3-\frac{3}{4}x$).
Mass $M=\iint_R (x+y)\,dA=\int_{x=0}^{4}\int_{y=0}^{3-3x/4}(x+y)\,dy\,dx$.

$$\int_0^{3-3x/4}(x+y)\,dy=\left[xy+\frac{y^2}{2}\right]_0^{3-3x/4}=x\left(3-\frac{3x}{4}\right)+\frac{1}{2}\left(3-\frac{3x}{4}\right)^2.$$

Compute: $3x-\frac{3x^2}{4}+\frac{1}{2}\left(9-\frac{9x}{2}+\frac{9x^2}{16}\right)=3x-\frac{3x^2}{4}+\frac{9}{2}-\frac{9x}{4}+\frac{9x^2}{32}$. Simplify linear term: $3x-\frac{9x}{4}=\frac{12x-9x}{4}=\frac{3x}{4}$. Quadratic term: $-\frac{3x^2}{4}+\frac{9x^2}{32}=-\frac{24x^2}{32}+\frac{9x^2}{32}=-\frac{15x^2}{32}$. Constant: $\frac{9}{2}$. Thus $M=\int_0^4\left(\frac{9}{2}+\frac{3x}{4}-\frac{15x^2}{32}\right)dx=\left[\frac{9}{2}x+\frac{3x^2}{8}-\frac{5x^3}{32}\right]_0^4$. At $x=4$: $\frac{9}{2}\cdot 4=18$, $\frac{3\cdot 16}{8}=6$, $-\frac{5\cdot 64}{32}=-10$. Sum $18+6-10=14$. Therefore $M=14$.

Exercise 03: Triple Integrals

1. $\iiint_V z\,dV$, $V=[0,1]^3$

$$\int_0^1\int_0^1\int_0^1 z\,dz\,dy\,dx=\int_0^1\int_0^1\left[\frac{z^2}{2}\right]_0^1 dy\,dx=\int_0^1\int_0^1\frac{1}{2}dy\,dx=\frac{1}{2}\cdot 1\cdot 1=\frac{1}{2}.$$

2. Volume bounded by $z=4-x^2-y^2$ and $z=0$ (cylindrical coordinates).
In cylindrical: $x=r\cos\theta$, $y=r\sin\theta$, $z=z$, $dV=r\,dr\,d\theta\,dz$.
Region: $0\leq\theta\leq 2\pi$, $0\leq r\leq 2$, $0\leq z\leq 4-r^2$.

$$V=\iiint dV=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{4-r^2} r\,dz\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^2 r(4-r^2)\,dr=2\pi\left[2r^2-\frac{r^4}{4}\right]_0^2.$$

$$2\pi\left(2\cdot 4-\frac{16}{4}\right)=2\pi(8-4)=2\pi\cdot 4=8\pi.$$

3. $\iiint_V (x^2+y^2+z^2)dV$ over $x^2+y^2+z^2\leq 1$ (spherical coordinates).
Spherical: $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$.
Jacobian matrix:

$$=\begin{pmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{pmatrix}$$

Determinant $=\rho^2\sin\phi$ (standard result). Thus $dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.
Region: $0\leq\rho\leq 1$, $0\leq\phi\leq\pi$, $0\leq\theta\leq 2\pi$. $x^2+y^2+z^2=\rho^2$.

$$\iiint\rho^2\,dV=\int_0^{2\pi}\int_0^{\pi}\int_0^1\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi\,d\phi\int_0^1\rho^4 d\rho.$$

$$=(2\pi)\cdot\big[-\cos\phi\big]_0^{\pi}\cdot\left[\frac{\rho^5}{5}\right]_0^1=2\pi\cdot(1+1)\cdot\frac{1}{5}=2\pi\cdot 2\cdot\frac{1}{5}=\frac{4\pi}{5}.$$

4. Center of gravity $\bar z$ of cone $\sqrt{x^2+y^2}\leq z\leq 1$, uniform density.
Cylindrical coordinates as above. Height $h=1$, radius $R=1$ at top.
Mass $M=\iiint dV=\int_0^{2\pi}\int_0^1\int_{z=r}^1 r\,dz\,dr\,d\theta=2\pi\int_0^1 r(1-r)dr=2\pi\left[\frac{r^2}{2}-\frac{r^3}{3}\right]_0^1=2\pi\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{\pi}{3}$.
First moment about xy-plane: $M_{xy}=\iiint z\,dV=\int_0^{2\pi}\int_0^1\int_{z=r}^1 z\cdot r\,dz\,dr\,d\theta=2\pi\int_0^1 r\left[\frac{z^2}{2}\right]_r^1 dr=\pi\int_0^1 r(1-r^2)dr$.

$$=\pi\int_0^1(r-r^3)dr=\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{\pi}{4}.$$

Thus $\bar{z} = \dfrac{M_{xy}}{M} = \dfrac{\pi/4}{\pi/3} = \dfrac{3}{4}$.

5. **Total charge** inside ball radius 2: $\rho_e(x,y,z) = 3 - \sqrt{x^2+y^2+z^2}$.
   Spherical coordinates: $\rho_e = 3 - \rho$, $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, $0 \le \rho \le 2$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

$$Q = \iiint (3-\rho)\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{2} (3-\rho)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

$$= \left(\int_0^{2\pi} d\theta\right) \left(\int_0^{\pi} \sin\phi \, d\phi\right) \left(\int_0^{2} (3\rho^2 - \rho^3) d\rho\right) = (2\pi) \cdot (2) \cdot \left[\rho^3 - \frac{\rho^4}{4}\right]_0^2.$$

$$= 4\pi \left(8 - \frac{16}{4}\right) = 4\pi(8-4) = 4\pi \cdot 4 = 16\pi \text{ C}.$$