Chapter 3 : Eigenvalues and Eigenvectors

As established in the previous chapters, a linear transformation from a finite-dimensional vector space to itself can be represented by a square matrix. Consequently, the study of the eigenvalues of a linear transformation is equivalent to the study of the eigenvalues of its matrix representation.

Qestion .

Why are we interested in the study of eigenvalues and eigenvectors ?

Answer :

Eigenvalues and eigenvectors of a matrix allow us to define a new basis, called an eigenbasis, in which the matrix takes a simpler form. In this basis, the matrix can often be written in a reduced form, such as a diagonal or triangular matrix. This simplification makes many computations and analyses easier to perform.

1 Definition

Let V be a vector space over a field F, and let $T : V \longrightarrow V$ be a linear transformation. The number λ is said to be an Eigenvalue if there exist a scalar $\lambda \in F$ and a vector $v \in V$ such that :

$$T(v) = \lambda v.$$

If A is the matrix representation of T, then we can write in equivalent way

$$Av = \lambda v.$$

In this case, if the vector $v \neq 0_V$, then is called an eigenvector of T (or of A) associated with the eigenvalue λ.

Example.

Consider a matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ of a linear transformation then

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

here $\lambda = 3$ is the eigenvalue of A and $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is its corresponding eigenvector. In the other hand, the vector $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is not an eigenvector, since $A\begin{pmatrix} 1 \\ 2 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$for all values of λ.

The equation $Av = \lambda v$ has a geometric interpretation : the vectors Av and v are collinear (i.e., parallel or anti-parallel). Moreover :

— if $\lambda > 0$, the vector v is stretched (its length increases).
— if $0 < \lambda < 0$, the vector v is contracted (its length decreases).
— if $\lambda < 0$, the vector v is reversed in direction (inverted), and its length is scaled by $|\lambda|$

1.1 Characteristic equation :

The equation $Av = \lambda v$ involves two unknowns : the scalar λ and the vector v. To determine these quantities, we rewrite the equation in the equivalent form

$$(A - \lambda I)\, v = 0,$$

wher I denotes the identity matrix of the same dimension as A. We are interested in nontrivial solutions, that is, vectors $v \neq 0$. Such solutions exist if and only if the matrix $A - \lambda I$ is singular, which is equivalent to the condition

$$det\,(A - \lambda I) = 0,$$

This equation is called the characteristic equation, and it allows us to determine the eigenvalues of the matrix A.

If A is an $n \times n$ matrix, then the characteristic euation gives rise to the characteristic polynomial of degree n, defined by

$$P_A(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots + \alpha_n \lambda^n.$$

The roots of this polynomial are precisely the eigenvalues of the matrix A.

Example 1.

Consider a matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

The characteristic equation of this matrix is given by

$$det\,(A - \lambda I) = (1-\lambda)^2 - 4 = 0$$

then we get the characteristic polynomial

$$P_A(\lambda) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$$

the roots of this polynomial are $\lambda_1 = 3$, $\lambda_2 = -1$ which represent the eigenvalues of the matrix A. The eigenvectors of the matrix A are determined for each eigenvalue as follows :

— For $\lambda_1 = 3$

Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$. The eigenvectors are obtained by solving the equation

$$(A - \lambda_1 I)\, v = 0$$

That is,

$$\begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This leads to the system

$$-2x + 2y = 0$$

which imply $x = y$

Hence, the eigenvectors corresponding to $\lambda = 3$ are of the form

$$v_1 = \begin{pmatrix} x \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x \neq 0$$

— For $\lambda_2 = -1$

Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$. The eigenvectors are obtained by solving the equation

$$(A - \lambda_2 I)\, v = 0$$

That is,

$$\begin{pmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This leads to the system

$$2x + 2y = 0$$

which imply $x = -y$

Hence, the eigenvectors corresponding to $\lambda = -1$ are of the form

$$v_2 = \begin{pmatrix} x \\ -x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad x \neq 0$$

Example 2.

Consider the three by three matrix

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ -2 & 4 & 2 \end{pmatrix}$$

— Find Eigenvalues :

$$det\,(B - \lambda I) = 0$$

calculate the determinant give :

$$(1 - \lambda)\,(1 - \lambda)\,(2 - \lambda) = 0$$

the solutions are

$$\lambda_1 = \lambda_2 = 1, \qquad \lambda_3 = 2$$

In this case, the matrix B has a repeated eigenvalue, which may lead to eigenvectors that are linearly dependent (i.e., not enough linearly independent eigenvectors).

— Find eigenvectors :

— for $\lambda_1 = \lambda_2 = 1$

Let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The equation $(B - \lambda_1 I)\,v = 0$ leads to the system :

$$\begin{aligned} z &= 0 \\ x &= 2y \end{aligned}$$

Hence, the eigenvectors corresponding to $\lambda_1 = \lambda_2 = 1$ are of the form

$$v_1 = v_2 = \begin{pmatrix} 2y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad y \neq 0$$

— for $\lambda_3 = 2$

The equation $(B - \lambda_3 I)\,v = 0$ leads to the following :

$$\begin{aligned} y &= z \\ x &= 2y \end{aligned}$$

Hence, the eigenvectors corresponding to $\lambda_3 = 2$ are of the form

$$v_3 = \begin{pmatrix} 2y \\ y \\ y \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad y \neq 0$$

1.2 Properties :

Let A be a $n \times n$ matrix :

— $tr(A) = \sum_{i=1}^{n} \lambda_i$

— $det(A) = \prod_{i=1}^{n} \lambda_i$

— If A is symmetric matrix $A = A^T$, then there exist n linearly independent eigenvectors of A.

— If A is singular, then A has a zero eigenvalue.

— if A is invertible and v is an eigenvector of A associated with the eigenvalue λ, then v is an eigenvector of A^{-1} associated with the eigenvalue $\frac{1}{\lambda}$

— If λ is an eigenvalue of A, then λ it is also an eigen value of A^T.

2 Diagonalisation of matices

The $n \times n$ matrix A is said to be diagonalisable if there exist an invertible matrix P such that

$$A = PDP^{-1}$$

Here, D is a diagonal matrix whose diagonal entries are the eigenvalues of A

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

and P is a matrix whose columns are the corresponding eigenvectors of A

$$P = \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & | & | \end{pmatrix}$$

The diagonalisation equation it can be reritten as follows :

$$P^{-1}AP = P^{-1}PDP^{-1}P$$
$$P^{-1}AP = \underbrace{P^{-1}P}_{I}D\underbrace{P^{-1}P}_{I}$$

and then we get

$$D = P^{-1}AP$$

In the case where the matrix of eigenvectors P is singular then A is not diagonalisable.

Example 1.

Let us diagonalize a matrix A such that

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

— Step 1 : Find the eigenvalues :

$$det\,(A - \lambda I) = \lambda^2 + 7\lambda - 10$$

Solve

$$\lambda^2 + 7\lambda - 10 = 0$$
$$\lambda_1 = 2, \quad \lambda_2 = 5$$

— Step 2 : Find the eigenvectors

For $\lambda_1 = 2$

$$(A - 2I)\,v = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

this system gives $y = -2x$

The corresponding eigenvecor for λ_1 : $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

For $\lambda_2 = 5$

$$(A - 5I)\, v = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

this system gives $y = x$

The corresponding eigenvecor for λ_2 : $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- Step 3 : Determine D, P and P^{-1}
  - The matrix of eigenvalues

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

  - The matrix of eigenvectors

$$P = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

  - The inverse matrix of eigenvectors

$$P^{-1} = \frac{1}{det(P)} adj(P) = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

- Step 4 : Check the diagonalization equation $A = PDP^{-1}$

Example 2.

Consider the the following matrix

$$Q = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

this matrix has the eigenvalues $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 5$. Since the eigenvalue 2 is repeated, the matrix does not necessarily admit two linearly independent eigenvectors associated with it. Consequently, the matrix formed by the eigenvectors is not invertible, and therefore the matrix Q is not diagonalizable.

3 Power of the matrix

If a matrix A is diagonalizable, then we can write A terms of its eigenvalues and eigenvectors matrices :

$$A = PDP^{-1}$$

and then we get the power of the matrix A as follows

$$\begin{aligned} A^k &= (PDP^{-1})^k \\ &= (PD\underbrace{P^{-1})(P}_{I}D\underbrace{P^{-1})(P}_{I}DP^{-1}) \cdots (PDP^{-1}) \\ &= PD^kP^{-1} \end{aligned}$$

where $k \in \mathbb{N}$.

Example :

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

Calculate A^{10}

The eigenvalues and eigenvectors of the matrix A :

- $\lambda_1 = 2, \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- $\lambda_2 = 3, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$
- $\lambda_3 = -1, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

The matrix A is diagonalizable (distinct eigenvalues) then we can write :

$$A = PDP^{-1}$$

where

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

Finding A^{10}

$$A^{10} = PD^{10}P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{10} & 0 & 0 \\ 0 & 3^{10} & 0 \\ 0 & 0 & (-1)^{10} \end{pmatrix} \begin{pmatrix} 2 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

4 Solution of linear differential equations

One of the fundamental applications of eigenvalues and eigenvectors lies in the solution of systems of differential equations, particularly linear systems that arise frequently in mathematics, physics, and engineering. These concepts provide a powerful framework for simplifying complex dynamical problems by transforming coupled differential equations into independent equations that are easier to solve.

Consider the following system of linear differential equations :

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\vdots$$
$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

This system is rewritten on the form $\frac{d}{dt}u = Au$, where

$$u = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

The solution of this linear system is given by :

$$u(t) = c_1 e^{-\lambda_1 t} v_1 + c_2 e^{-\lambda_2 t} v_2 + \cdots + c_n e^{-\lambda_n t} v_n$$

where $c_1, \ c_2, \ \dots, c_n$are constants, the scalars $\lambda_1, \ \lambda_2, \dots \lambda_n$ are the eigenvalues of A, and the vectors $v_1, \ v_2 \cdots v_n$ are the eigenvectors of A.

Example :

The system of linear differential equation :

$$\frac{dx}{dt} = 4x - 5y$$
$$\frac{dy}{dt} = x - 2y$$

This system it can be rewritten as $\frac{d}{dt}u = Au$, where $u = \begin{pmatrix} x \\ y \end{pmatrix}$and $A = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$.
The form of the solution is given by :

$$u(t) = c_1 e^{-\lambda_1 t} v_1 + c_2 e^{-\lambda_2 t} v_2$$

To solve this system first, we find eigenvalues and eigenvectors of A and then we replace them in the previouse formula.

— The eigenvalues and eigenvectors of A :

$$\lambda_1 = 3, \ v_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -1, \ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, we get

$$u(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-3t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The constants $c_1$and c_2 are determined according to the initial conditions.