Analysis II: Solutions of Tutorial Exercise Sheet 2

Hocine RANDJI

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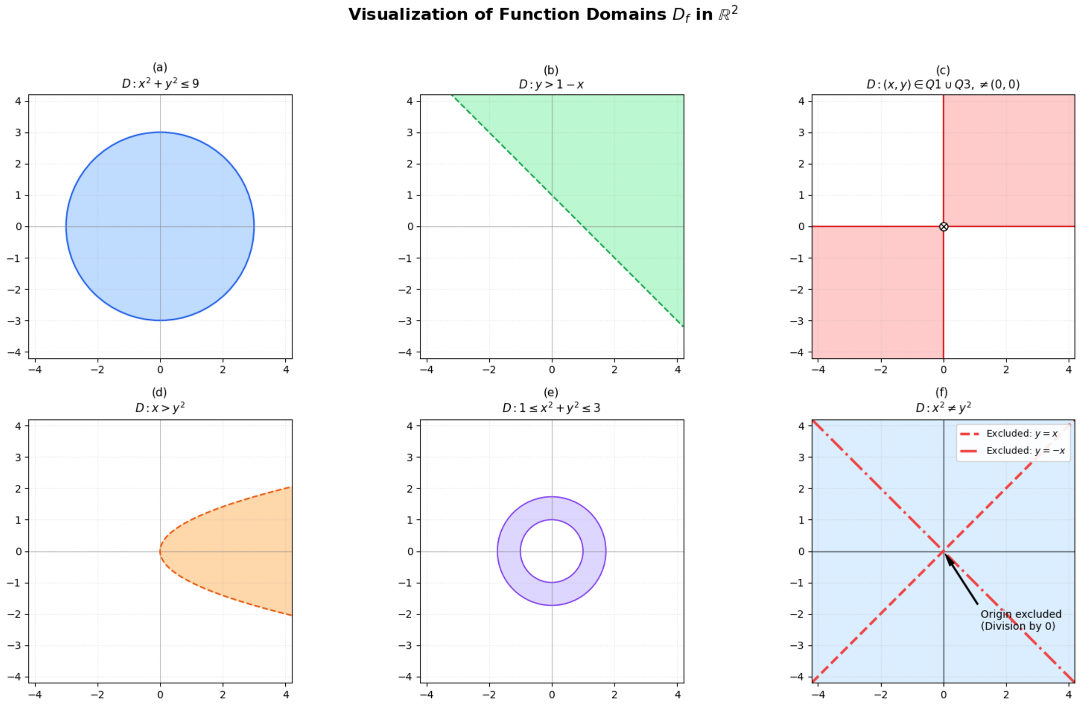
Exercise 01: Domain and Limits

1. (a) $f(x,y)=\sqrt{9-x^2-y^2}$ defined for $x^2+y^2\le 9$. Closed disk of radius 3 centered at the origin.
   (b) $f(x,y)=\ln(x+y-1)$ defined for $x+y>1$. Open half-plane above the line $y=1-x$.
   (c) $f(x,y)=\arcsin\frac{x-y}{x+y}$ defined for $x+y\neq 0$ and $-1\le\frac{x-y}{x+y}\le 1\Rightarrow$ first and third quadrants (axes included, origin excluded). Exclude the line $y=-x$.
   (d) $f(x,y)=\frac{1}{\sqrt{x-y^2}}$ defined for $x>y^2$. Open region to the right of the parabola $x=y^2$.
   (e) $f(x,y)=\arccos(x^2+y^2-2)$ defined for $1\le x^2+y^2\le 3$. Closed annulus between radii 1 and $\sqrt{3}$.
   (f) $f(x,y)=\frac{1}{x^2-y^2}$ defined for $x^2\neq y^2\Rightarrow y\neq x$ and $y\neq -x$. Entire plane except these two lines.



2. (a) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$

   Path 1: $y=x^2\implies\frac{x^2\cdot x^2}{x^4+(x^2)^2}=\frac{x^4}{2x^4}=\frac{1}{2}$

   Path 2: $y=x\implies\frac{x^2\cdot x}{x^4+x^2}=\frac{x^3}{x^2(x^2+1)}=\frac{x}{x^2+1}\to 0$ Different limits $\Rightarrow$ no limit.

   (b) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{x^2+y^2}$

   Path 1: $y=0\implies\frac{x^2}{x^2}=1$

   Path 2: $x=0\implies\frac{-y^2}{y^2}=-1$ Different limits $\Rightarrow$ no limit.

3. Use polar coordinates: $x=r\cos\theta$, $y=r\sin\theta$. As $(x,y)\to(0,0)$, $r\to 0$.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2+y^2}$

$$\frac{(r\cos\theta)^2(r\sin\theta)^2}{r^2\cos^2\theta+r^2\sin^2\theta}=\frac{r^4\cos^2\theta\sin^2\theta}{r^2}=r^2\cos^2\theta\sin^2\theta\to 0$$

   Explanation: $|\cos^2\theta\sin^2\theta|\le 1$ and $r^2\to 0$, so the product tends to 0.

   (b) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^3y}{x^2+y^2}$

$$\frac{(r\cos\theta)^3(r\sin\theta)}{r^2}=\frac{r^4\cos^3\theta\sin\theta}{r^2}=r^2\cos^3\theta\sin\theta\to 0$$

Explanation: $|\cos^3\theta\sin\theta| \le 1$ and $r^2 \to 0$, so the product tends to 0.

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

$$\frac{(r\cos\theta)(r\sin\theta)^2}{r^2} = \frac{r^3\cos\theta\sin^2\theta}{r^2} = r\cos\theta\sin^2\theta \to 0$$

Explanation: $|\cos\theta\sin^2\theta| \le 1$ and $r \to 0$, so the product tends to 0.

All limits equal 0 because the trigonometric factor is bounded while the power of r vanishes.

Exercise 02: Partial Derivatives and Differentiability

1. Compute f_x, f_y, f_{xx}, f_{xy}:

   (a) $f(x,y) = \sin(xy) + e^{x^2y}$

$$\begin{aligned} f_x &= y\cos(xy) + 2xy\,e^{x^2y} \\ f_y &= x\cos(xy) + x^2e^{x^2y} \\ f_{xx} &= -y^2\sin(xy) + 2ye^{x^2y} + 4x^2y^2e^{x^2y} \\ f_{xy} &= \cos(xy) - xy\sin(xy) + 2xe^{x^2y} + 2x^3ye^{x^2y} \end{aligned}$$

   (b) $f(x,y) = \arctan(y/x)$

$$\begin{aligned} f_x &= \frac{-y}{x^2+y^2}, \quad f_y = \frac{x}{x^2+y^2} \\ f_{xx} &= \frac{2xy}{(x^2+y^2)^2}, \quad f_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}$$

   (c) $f(x,y) = x^y \ (x>0)$

$$\begin{aligned} f_x &= yx^{y-1}, \quad f_y = x^y\ln x \\ f_{xx} &= y(y-1)x^{y-2}, \quad f_{xy} = x^{y-1} + yx^{y-1}\ln x \end{aligned}$$

2. Show $f(x,y) = \ln(x^2+y^2)$ is harmonic: $f_{xx} + f_{yy} = 0$.

$$\begin{aligned} f_x &= \frac{2x}{x^2+y^2}, \quad f_{xx} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} \\ f_y &= \frac{2y}{x^2+y^2}, \quad f_{yy} = \frac{2(x^2-y^2)}{(x^2+y^2)^2} \\ f_{xx}+f_{yy} &= \frac{2(y^2-x^2)+2(x^2-y^2)}{(x^2+y^2)^2} = 0 \end{aligned}$$

3. Find the tangent plane to $z = \sqrt{x^2+y^2}$ at $(3,4,5)$.
   Let $z = f(x,y) = \sqrt{x^2+y^2}$. Then

$$f_x = \frac{x}{\sqrt{x^2+y^2}}, \quad f_y = \frac{y}{\sqrt{x^2+y^2}}$$

   At $(3,4)$: $f_x(3,4) = \frac{3}{5}$, $f_y(3,4) = \frac{4}{5}$. Tangent plane:

$$z - 5 = \frac{3}{5}(x-3) + \frac{4}{5}(y-4) \quad \Rightarrow \quad 3x + 4y - 5z = 0$$

4. Show $f(x,y)=\sqrt{|xy|}$ is not differentiable at $(0,0)$ despite $f_x(0,0)=f_y(0,0)=0$.
   Check definition: Differentiability would require
$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-f_x(0,0)h-f_y(0,0)k}{\sqrt{h^2+k^2}}=0$$
   Here the expression becomes $\frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}$. Take $h=k$:
$$\frac{\sqrt{|h^2|}}{\sqrt{2h^2}}=\frac{|h|}{\sqrt{2}|h|}=\frac{1}{\sqrt{2}}\neq 0$$
   Thus the limit is not zero, so f is not differentiable at $(0,0)$.

Exercise 03: Chain Rule and Taylor

1. $z=e^{u-v}$, $u=x^2+y$, $v=x-y^2$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
   Chain rule:
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=e^{u-v}(2x)+(-e^{u-v})(1)=e^{u-v}(2x-1)$$
   For y, note that $\frac{\partial z}{\partial v}=-e^{u-v}$ and $\frac{\partial v}{\partial y}=-2y$:
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=e^{u-v}(1)+(-e^{u-v})(-2y)=e^{u-v}(1+2y)$$
   Substitute back $u-v=x^2-x+y+y^2$.
2. Ideal gas: $P=nRT/V$, with $T=T(t)$, $V=V(t)$. Find dP/dt by chain rule.
$$\frac{dP}{dt}=\frac{\partial P}{\partial T}\frac{dT}{dt}+\frac{\partial P}{\partial V}\frac{dV}{dt}=\frac{nR}{V}\frac{dT}{dt}-\frac{nRT}{V^2}\frac{dV}{dt}$$
3. Recall the second-order Taylor expansion at (a,b):
$$f(x,y)=f(a,b)+\nabla f\cdot\begin{pmatrix}h\\k\end{pmatrix}+\frac{1}{2}\begin{pmatrix}h & k\end{pmatrix}H\begin{pmatrix}h\\k\end{pmatrix}+O(r^3),$$
   where $h=x-a,\ k=y-b,\ r=\sqrt{h^2+k^2}$, and
$$H=\begin{pmatrix}f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{pmatrix}.$$
   For all expansions below, $(a,b)=(0,0)$, so $h=x,\ k=y$.
   For $f(x,y)=\cos x\sin y$
$$\begin{aligned} f(0,0)&=0,\\ \nabla f&=\big(f_x(0,0),\ f_y(0,0)\big)=\big(-\sin 0\sin y,\ \cos x\cos y\big)\big|_{(0,0)}=(0,1),\\ H&=\begin{pmatrix}f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{pmatrix}_{(0,0)}=\begin{pmatrix}-\cos x\sin y & -\sin x\cos y\\ -\sin x\cos y & -\cos x\sin y\end{pmatrix}_{(0,0)}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.\end{aligned}$$
   Hence
$$f(x,y)=0+(0,1)\cdot\begin{pmatrix}x\\y\end{pmatrix}+\frac{1}{2}\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+O(r^3)=y+O(r^3).$$
4. $f(x,y)=e^x\cos y$
$$\begin{aligned} f(0,0)&=1,\\ \nabla f&=\big(e^x\cos y,\ -e^x\sin y\big)\big|_{(0,0)}=(1,0),\\ H&=\begin{pmatrix}e^x\cos y & -e^x\sin y\\ -e^x\sin y & -e^x\cos y\end{pmatrix}_{(0,0)}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.\end{aligned}$$
   Thus
$$f(x,y)=1+(1,0)\cdot\begin{pmatrix}x\\y\end{pmatrix}+\frac{1}{2}\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+O(r^3)=1+x+\frac{x^2}{2}-\frac{y^2}{2}+O(r^3).$$

5. $f(x,y) = \ln(1+x+2y)$

$$\begin{aligned}
f(0,0) &= 0, \\
\nabla f &= \left(\frac{1}{1+x+2y}, \frac{2}{1+x+2y}\right)\Big|_{(0,0)} = (1,2), \\
f_{xx} &= -\frac{1}{(1+x+2y)^2}, \quad f_{yy} = -\frac{4}{(1+x+2y)^2}, \quad f_{xy} = f_{yx} = -\frac{2}{(1+x+2y)^2}, \\
H_{(0,0)} &= \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
f(x,y) &= 0 + (1,2)\cdot\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + O(r^3) \\
&= x + 2y + \frac{1}{2}\left(-x^2 - 4xy - 4y^2\right) + O(r^3) \\
&= x + 2y - \frac{1}{2}x^2 - 2xy - 2y^2 + O(r^3).
\end{aligned}$$

Exercise 04: Optimization and Lagrange

1. (a) $f(x,y) = x^4 + y^4 - 4xy$

Critical points:

$$\begin{cases} f_x = 4x^3 - 4y = 0 \Rightarrow y = x^3 \\ f_y = 4y^3 - 4x = 0 \Rightarrow x = y^3 \end{cases} \quad \Rightarrow x = (x^3)^3 = x^9 \Rightarrow x(x^8-1) = 0$$

$$x = 0,\ 1,\ -1 \Rightarrow (0,0),\ (1,1),\ (-1,-1)$$

Second derivatives:

$$f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = -4, \quad D = \det H = 144x^2y^2 - 16$$

Classification:

(x,y)	D	Type
$(0,0)$	$-16 < 0$	Saddle point
$(1,1)$	$128 > 0,\ f_{xx} = 12 > 0$	Local minimum
$(-1,-1)$	$128 > 0,\ f_{xx} = 12 > 0$	Local minimum

(b) $f(x,y) = x^2 + xy + y^2 - 3x$

Critical point:

$$\begin{cases} f_x = 2x + y - 3 = 0 \\ f_y = x + 2y = 0 \Rightarrow x = -2y \end{cases} \quad \Rightarrow 2(-2y) + y - 3 = 0 \Rightarrow -3y = 3$$

$$y = -1,\ x = 2 \Rightarrow (2,-1)$$

Second derivatives:

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1, \quad D = \det H = (2)(2) - 1^2 = 3 > 0$$

Classification:

$$(2,-1):\ D = 3 > 0,\ f_{xx} = 2 > 0 \Rightarrow \text{Local minimum}$$

2. **Box surface area**

Minimize $f(x,y,z) = 2(xy + xz + yz)$ subject to $\phi(x,y,z) = xyz - 32 = 0$.

Lagrangian: $h(x,y,z,\lambda) = 2(xy + xz + yz) + \lambda(xyz - 32)$.

$\nabla h = 0$:

$$\begin{cases} h_x & = 2(y+z) + \lambda yz = 0 \quad (1) \\ h_y & = 2(x+z) + \lambda xz = 0 \quad (2) \\ h_z & = 2(x+y) + \lambda xy = 0 \quad (3) \\ xyz & = 32 \quad (4) \end{cases}$$

From (1) and (2): $\lambda = -\frac{2(y+z)}{yz} = -\frac{2(x+z)}{xz} \Rightarrow x(y+z) = y(x+z) \Rightarrow xz = yz \Rightarrow x = y$.

From (2) and (3): $\lambda = -\frac{2(x+z)}{xz} = -\frac{2(x+y)}{xy} \Rightarrow y(x+z) = z(x+y) \Rightarrow xy = xz \Rightarrow y = z$.

Thus $x = y = z$. Substitute in (4): $x^3 = 32 \Rightarrow x = \sqrt[3]{32} = 2\sqrt[3]{4}$.

Minimum surface area: $f_{\min} = 6x^2 = 6(32)^{2/3} = 6 \cdot 2^{10/3} \approx 60.48\ \mathrm{m}^2$.

3. **Minimize $x^2 + y^2$ subject to $2x + 3y = 12$**

   Minimize $f(x,y) = x^2 + y^2$ subject to $\phi(x,y) = 2x + 3y - 12 = 0$.

   Lagrangian: $h(x,y,\lambda) = x^2 + y^2 + \lambda(2x + 3y - 12)$.

   $\nabla h = 0$:

   $$\begin{cases} h_x = 2x + 2\lambda = 0 \Rightarrow x = -\lambda \quad (1) \\ h_y = 2y + 3\lambda = 0 \Rightarrow y = -\frac{3\lambda}{2} \quad (2) \\ \phi = 2x + 3y - 12 = 0 \quad (3) \end{cases}$$

   Substitute (1) and (2) into (3):

   $$2(-\lambda) + 3\left(-\frac{3\lambda}{2}\right) = 12 \Rightarrow -2\lambda - \frac{9\lambda}{2} = 12 \Rightarrow -\frac{13\lambda}{2} = 12 \Rightarrow \lambda = -\frac{24}{13}.$$

   Then $x = -\left(-\frac{24}{13}\right) = \frac{24}{13}$, $y = -\frac{3}{2}\left(-\frac{24}{13}\right) = \frac{36}{13}$.

   Minimum value: $f_{\min} = \left(\frac{24}{13}\right)^2 + \left(\frac{36}{13}\right)^2 = \frac{576+1296}{169} = \frac{1872}{169}$.

4. **Square maximizes area for fixed perimeter**

   Maximize $f(x,y) = xy$ subject to $\phi(x,y) = 2x + 2y - P = 0$ (P constant).

   Lagrangian: $h(x,y,\lambda) = xy + \lambda(2x + 2y - P)$.

   $\nabla h = 0$:

   $$\begin{cases} h_x = y + 2\lambda = 0 \Rightarrow y = -2\lambda \quad (1) \\ h_y = x + 2\lambda = 0 \Rightarrow x = -2\lambda \quad (2) \\ 2x + 2y = P \quad (3) \end{cases}$$

   From (1) and (2): $x = y$. Substitute into (3): $2x + 2x = P \Rightarrow 4x = P \Rightarrow x = \frac{P}{4}$, $y = \frac{P}{4}$.

   Thus the rectangle is a square. Since the constraint is compact and the objective continuous, this critical point gives the maximum area.

5. **Current division / heat minimization**

   Minimize $f(I_1, I_2, I_3) = R_1 I_1^2 + R_2 I_2^2 + R_3 I_3^2$ subject to $\phi(I_1, I_2, I_3) = I_1 + I_2 + I_3 - I = 0$.

   Lagrangian: $h(I_1, I_2, I_3, \lambda) = R_1 I_1^2 + R_2 I_2^2 + R_3 I_3^2 - \lambda(I_1 + I_2 + I_3 - I)$.

   $\nabla h = 0$:

   $$\begin{cases} h_{I_1} = 2R_1 I_1 - \lambda = 0 \Rightarrow h_1 = \frac{\lambda}{2R_1} \\ h_{I_2} = 2R_2 I_2 - \lambda = 0 \Rightarrow I_2 = \frac{\lambda}{2R_2} \\ h_{I_3} = 2R_3 I_3 - \lambda = 0 \Rightarrow I_3 = \frac{\lambda}{2R_3} \\ I_1 + I_2 + I_3 = I \ (*) \end{cases}$$

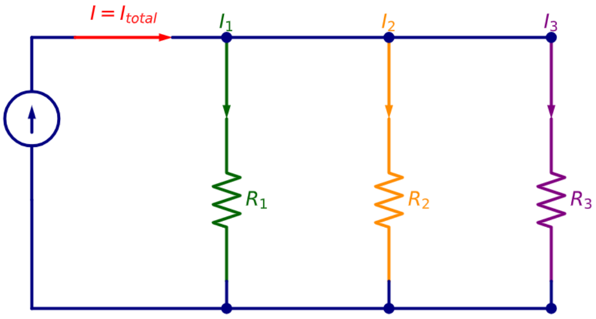
   Substitute into the sum $(*)$:

   $$\frac{\lambda}{2R_1} + \frac{\lambda}{2R_2} + \frac{\lambda}{2R_3} = I \Rightarrow \frac{\lambda}{2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = I.$$

Let $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \ \Rightarrow\ \frac{\lambda}{2} = IR_{eq}$.

Thus $I_1 = I\frac{R_{eq}}{R_1}, \quad I_2 = I\frac{R_{eq}}{R_2}, \quad I_3 = I\frac{R_{eq}}{R_3}$.

Interpretation: $I_1R_1 = I_2R_2 = I_3R_3 = IR_{eq} = V$ (voltage drop identical).



Exercise 05: Miscellaneous Problems

1. **(Implicit relations and thermodynamics)**

   Let $F(P,V,T) = 0$. Total differential: $dF = F_P dP + F_V dV + F_T dT = 0$, where $F_P = \left(\frac{\partial F}{\partial P}\right)_{V,T}$, $F_V = \left(\frac{\partial F}{\partial V}\right)_{P,T}$, $F_T = \left(\frac{\partial F}{\partial T}\right)_{P,V}$.

   To find partial derivatives, we fix one variable:

   1. Fix V $(dV = 0)$: $F_P dP + F_T dT = 0 \Rightarrow \frac{\partial P}{\partial T}\big|_V = -\frac{F_T}{F_P}$.

   2. Fix P $(dP = 0)$: $F_V dV + F_T dT = 0 \Rightarrow \frac{\partial T}{\partial V}\big|_P = -\frac{F_V}{F_T}$.

   3. Fix T $(dT = 0)$: $F_P dP + F_V dV = 0 \Rightarrow \frac{\partial P}{\partial V}\big|_T = -\frac{F_V}{F_P}, \ \frac{\partial V}{\partial P}\big|_T = -\frac{F_P}{F_V}$.

   (a) $\frac{\partial P}{\partial T}\big|_V \frac{\partial T}{\partial V}\big|_P = \left(-\frac{F_T}{F_P}\right)\left(-\frac{F_V}{F_T}\right) = \frac{F_V}{F_P} = -\ \frac{\partial P}{\partial V}\big|_T$.

   (b) $\frac{\partial P}{\partial T}\big|_V \frac{\partial T}{\partial V}\big|_P \frac{\partial V}{\partial P}\big|_T = \left(\frac{F_V}{F_P}\right)\left(-\frac{F_P}{F_V}\right) = -1$.

2. **(Laplace equation)**

   Let $u(x,y) = e^x \cos y$.

   First derivatives: $u_x = e^x \cos y$, $u_y = -e^x \sin y$.

   Second derivatives: $u_{xx} = e^x \cos y$, $u_{yy} = -e^x \cos y$.

   Laplace equation: $u_{xx} + u_{yy} = e^x \cos y + (-e^x \cos y) = 0$.

   $\therefore$ u satisfies Laplace's equation.

3. **(Error propagation)**

   $V = \pi r^2 h$. Total differential: $dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$.

   $\frac{\partial V}{\partial r} = 2\pi rh$, $\frac{\partial V}{\partial h} = \pi r^2$.

   $\Rightarrow dV = 2\pi rh\,dr + \pi r^2\,dh$.

   Divide by $V = \pi r^2 h$: $\dfrac{dV}{V} = \dfrac{2\pi rh\,dr}{\pi r^2 h} + \dfrac{\pi r^2\,dh}{\pi r^2 h} = 2\,\dfrac{dr}{r} + \dfrac{dh}{h}$.

   Relative errors: $\left|\dfrac{dr}{r}\right| = 1\% = 0.01$, $\left|\dfrac{dh}{h}\right| = 0.01$.

   Maximum relative error: $\left|\dfrac{dV}{V}\right|_{\max} = 2(0.01) + 0.01 = 0.03 = 3\%$.

4. **(Electrostatics)**

   Given $V = \dfrac{k}{r}$, $r = \sqrt{x^2 + y^2 + z^2}$, $k = \dfrac{q}{4\pi\varepsilon_0}$.

   (1) $\vec{E} = -\nabla V = -\left(\dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y}, \dfrac{\partial V}{\partial z}\right)$.

$\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial V}{\partial x} = k\left(-\dfrac{1}{r^2}\right)\dfrac{\partial r}{\partial x} = -\dfrac{kx}{r^3}$.

Similarly $\dfrac{\partial V}{\partial y} = -\dfrac{ky}{r^3}$, $\dfrac{\partial V}{\partial z} = -\dfrac{kz}{r^3}$.

$\Rightarrow \nabla V = -\dfrac{k}{r^3}(x, y, z) = -\dfrac{k}{r^3}\vec{r}.$

$\therefore \vec{E} = -\nabla V = \dfrac{k}{r^3}\,\vec{r}.$

(2) $\Delta V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2}$.

$\dfrac{\partial V}{\partial x} = -kx\,r^{-3}$.

$\dfrac{\partial^2 V}{\partial x^2} = -k\left[r^{-3} + x(-3)r^{-4}\dfrac{x}{r}\right] = -k\left[r^{-3} - 3x^2 r^{-5}\right] = -k\,r^{-5}(r^2 - 3x^2).$

By symmetry: $\dfrac{\partial^2 V}{\partial y^2} = -k\,r^{-5}(r^2 - 3y^2)$, $\dfrac{\partial^2 V}{\partial z^2} = -k\,r^{-5}(r^2 - 3z^2)$.

$\Delta V = -kr^{-5}\Big[(r^2 - 3x^2) + (r^2 - 3y^2) + (r^2 - 3z^2)\Big] = -kr^{-5}\big[3r^2 - 3(x^2 + y^2 + z^2)\big]$

$= -kr^{-5}(3r^2 - 3r^2) = 0$, $(x, y, z) \neq (0, 0, 0)$.