Basic concepts in probability

1 Set Notation

You may omit this section if you are familiar with these concepts.

A set is a collection of objects.

We often specify a set by listing its members, or **elements**, in parentheses like this {}.

For example $A = \{2, 4, 6, 8\}$ means that A is the set consisting of numbers 2,4,6,8.

We could also write A ={even numbers less than 9}.

The **union** of A and B is the set of elements which belong to A or to B (or both) and can be written as $A \cup B$.

The **intersection** of A and B is the set of elements which belong to both A and B, and can be written as $A \cap B$.

The **complement** of A, frequently denoted by $\overline{A}$, is the set of all elements which do not belong to A. In making this definition we assume that all elements we are thinking about belong to some larger set U, which we call the **universal** set.

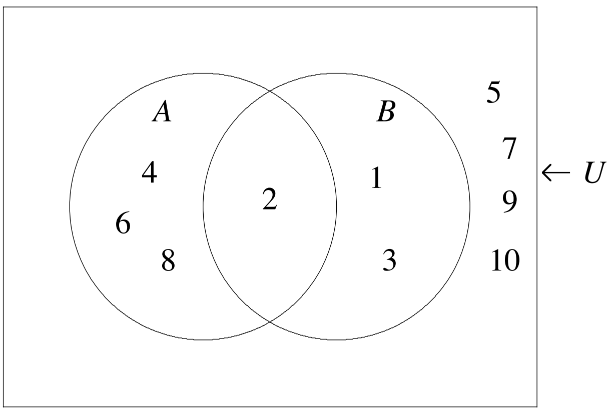
The **empty set**, written $\emptyset$ or {}, means the set with no elements in it.

A set C is a **subset** of A if **all** the elements in C are also in A.

For example, let

$$\begin{aligned} U &= \{\text{all positive numbers} \leq 10\} \\ A &= \{2, 4, 6, 8\} \\ B &= \{1, 2, 3\} \\ C &= \{6, 8\} \end{aligned}$$

Sets A, B and U may be represented in a Venn Diagram as follows:



A intersection B, $A \cap B$, is shown in the Venn diagram by the overlap of the sets A and B, $A \cap B = \{2\}$.

The union of the sets A and B, $A \cup B$, is the set of elements that are in $A = \{2, 4, 6, 8\}$ together with the elements that are in $B = \{1, 2, 3\}$ including each element once only.

So, $A \cup B = \{1, 2, 3, 4, 6, 8\}$.

The complement of A is the set $\overline{A}$ is contains all the elements in U which are not in A. So, $\overline{A} = \{1, 3, 5, 7, 9, 10\}$.

C is a subset of A as every element in $C = \{6, 8\}$ is also in $A = \{2, 4, 6, 8\}$.

2 Finite Equiprobable Spaces

In loose terms, we say that the probability of something happening is $\frac{1}{4}$, if, when the experiment is repeated often under the same conditions, the stated result occurs 25% of the time.

For the moment, we will confine our discussion to cases where there are a finite number of equally likely outcomes.

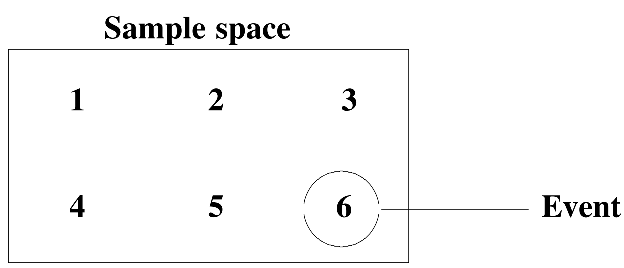
For example, if a coin is tossed, there are two equally likely outcomes: heads (H) or tails (T). If a die is tossed, there are six equally likely outcomes: 1,2,3,4,5,6.

Some Notation

The set of all possible outcomes of the given experiment is called the **sample space**. An **event** is a subset of a sample space.

Calculating Probabilities

Look again at the example of rolling a six faced die. The possible outcomes in this experiment are 1,2,3,4,5,6, so the sample space is the set {1,2,3,4,5,6}. The 'event' of 'getting a 6' is the subset {6}. We represent this in the following diagram.



There are six possibilities in the sample space and only one of these corresponds to getting a 6, so the probability of getting a 6 when you roll a die is $\frac{1}{6}$.

We say that the probability of an event A occuring is

$$P(A) = \frac{\text{Number of elements in } A}{\text{Total number of elements in the sample space}}$$

Example

If a fair coin is tossed, it is clear from our definition of probability above that P (obtaining a head) $= \frac{1}{2}$.

Example

A card is selected at random from a pack of 52 cards. Let $A =$ 'the card is a heart' and $B =$ 'the card is an ace'.

Find $P(A),\ P(B)$.

Solution

$P(A) = \frac{13}{52}$ since there are 13 hearts in the pack. $P(B) = \frac{4}{52}$ since there are 4 aces in the pack.

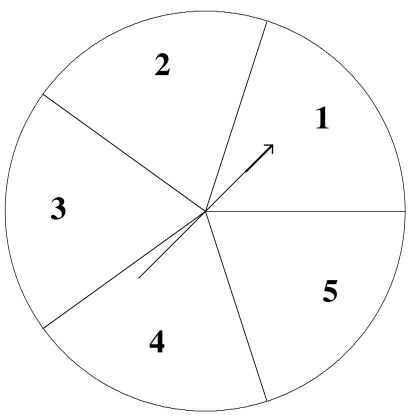
To calculate the probability of an event, we simply need to find out the total number of possible outcomes of an experiment and the number of outcomes which correspond to the **given** event.

Exercise 1

What are your chances of winning a raffle in which 325 tickets have been sold, if you have one ticket?

Exercise 2

A cursor is spun on a disc divided into five equal sectors as shown below. The position of the pointer is noted. (If it is on a line the cursor is spun again.)



Let A be the event 'pointer is in the first sector' and B the event 'pointer is in the 2nd or 4th sector'.

Find $P(A)$, $P(B)$.

Example

Consider the following problem. Two coins are tossed. Let A be the event 'two heads are obtained', and, B be the event 'one head and one tail is obtained'.

Find $P(A),\ P(B)$.

Solution

The sample space={HH, HT, TH, TT}

A={HH}

B={HT, TH}.

Since there are 4 outcomes in the sample space.

$P(A) = \dfrac{1}{4}$

$P(B) = \dfrac{2}{4} = \dfrac{1}{2}$

Notice that HT and TH must be regarded as **different** outcomes.

Often we can conveniently represent the possible outcomes on a diagram and count directly. We will also develop some techniques and rules to assist in our calculations.

Now let us see what happens in reality. Try the following experiment:

Roll a die 50 times and record the number of each of the outcomes 1,2,3,4,5,6.

Continue rolling and record the number of each outcome after 100 rolls. Now record the number after 200 rolls. Find the **relative frequency** of each outcome after 50, 100 and 200 rolls.

For example calculate $\dfrac{\text{the number of times `1' occurs}}{\text{total number of rolls}}$.

Does it get closer to $\frac{1}{6}$, i.e. 0.17?

3 Complementary Events

If an event is a **certainty**, then its probability is one.

In common language we often say it is 100% certain (which is the same thing).

For example, in the coin tossing experiment, let C be the event 'obtaining a head or a tail'.

The sample space is {H, T}. The event is {H, T}.

So $P(C) = \frac{2}{2} = 1$.

Example

If a normal die is rolled, what is the probability that the number showing is less than 7?

Solution

Sample space = {1,2,3,4,5,6}

Event = {1,2,3,4,5,6}

Hence the probability (number is less than 7) $= \frac{6}{6} = 1$.

If an event is **impossible**, then its probability is zero.

Example

Find the probability of throwing an 8 on a normal die.

Here there are **no** possible outcomes in the event.

i.e. Sample space = $\{1,2,3,4,5,6\}$

Event = $\{\}$, i.e. the empty set.

Hence the probability of throwing an 8 is $\frac{0}{6} = 0$.

If the event is neither impossible nor certain, then clearly its **probability is between 0 and 1**.

Two events are **complementary** if they cannot occur at the same time and they make up the whole sample space.

Example

When a coin is tossed, the sample space is $\{H, T\}$ and the events H = 'obtain a head' and T = 'obtain a tail' are complementary.

If we calculate the probabilities we find that

$P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$ and $P(H) + P(T) = 1$.

Example

A die is rolled. Let A be the event 'a number less than 3 is obtained' and let B be the event 'a number of 3 or more is obtained'.

Then $P(A) = \frac{2}{6}$, and $P(B) = \frac{4}{6}$.

So that $P(A) + P(B) = 1$.

We have illustrated following law.

If two events are **complementary**, then their **probabilities add up to 1**.

Example

A marble is drawn at random from a bag containing 3 red, 2 blue, 5 green and 1 yellow marble. What is the probability that it is not green?

Solution

There are two ways of doing this problem.

Method A:

We can work out the probability that the marble is green:

$P(G) = \frac{5}{11}$.

Since a marble is either green or not green, the probability that it is not green,

$P(\overline{G}) = 1 - \frac{5}{11} = \frac{6}{11}$.

Method B:

Alternatively, we can find the probability that the marble is red, blue or yellow which is $\frac{6}{11}$.

Exercise 3

Three tulip bulbs are planted in a window box. Find the probability that at least one will flower if the probability that all will fail to flower is $\frac{1}{8}$.

Sometimes calculations are made easier by using complementary events.

4 Mutually Exclusive Events

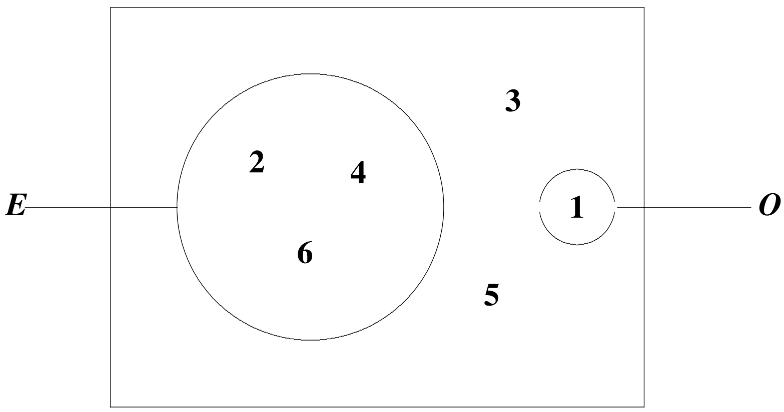
Two events are **incompatible, disjoint** or **mutually exclusive** when the occurence of one precludes the occurrence of the other, i.e. they can not occur at the same time. For example, we can never have the head side and the tail side of a coin face up at the same time.

Example

Suppose a die is tossed.

Then the events E = 'obtaining an even number'

and O = 'obtaining a one' are mutually exclusive.



On our diagram the events do not intersect.

$$\begin{aligned}\text{Notice that } P(\text{throwing an even number or one}) &= P(1,2,4,6)\\ &= \frac{4}{6}\\ &= P(E)+P(O).\end{aligned}$$

Example

What is the probability of drawing a heart or spade from a pack of 52 cards when one card is drawn at random?

Solution

$P(\text{heart}) = \frac{13}{52}$

$P(\text{spade}) = \frac{13}{52}$

$P(\text{heart or spade}) = \frac{26}{52}$ since 26 of the cards are either hearts or spades.

Notice $P(\text{heart or spade}) = P(\text{heart}) + P$ (spade).

We may now state the **addition law** for **mutually exclusive events**.

If two events A and B are mutually exclusive, the probability of A or B happening, denoted $P(A \cup B)$, is:

$$P(A \cup B) = P(A) + P(B).$$

Exercise 4

A number is selected at random from the integers 2 to 25. Find the probability that it is:

(a) a perfect square;

(b) a prime number;

(c) a prime number or perfect square.

Example

What is the flaw in the following argument?

'Seventy percent of first year science students study mathematics. Thirty percent of first year science students study chemistry. If a first year science student is selected at random, the probability that the student is taking maths is $\frac{70}{100}$, the probability that the student is taking chemistry is $\frac{30}{100}$, hence the probability that the student is taking maths or chemistry is $\frac{70}{100} + \frac{30}{100} = 1$ (i.e. a certainty).'

Solution

The two events are not mutually exclusive, therefore we cannot add the probabilities.

That is, to count all the students doing maths and/or chemistry, we need to count all the maths students, all the chemistry students, and subtract from this the number of students who were counted twice because they were in both classes.

If $M \cup C$ means that at least one of M or C occurs and $M \cap C$ means that both M and C occur, then

$$P(M \cup C) = P(M) + P(C) - P(M \cap C).$$

To Summarise:

For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
If A and B are mutually exclusive then $P(A \cap B) = 0$, and A and B cannot happen together, so that $P(A \cup B) = P(A) + P(B)$.

Exercise 5

A maths class consists of 14 women and 16 men. Of these, 12 of the men and half of the women study computer science. A person is chosen at random from the class. Find the probability that the person selected is:

(a) a woman

(b) studying computer science;

(c) a woman who is studying computer science;

(d) a woman or is taking computer science.

Exercise 6

A bag of marbles contains 23 Tiger's Eyes, 17 Rainbows and 5 Pearls.
One marble is drawn at random.
Denote by:
T the event 'a Tiger's Eye is drawn';
R the event 'a Rainbow is drawn';
P the event 'a Pearl is drawn'.
Describe the following events in words and find their probabilities:

(a) $R \cup T$,

(b) $R \cap T$,

(c) $T \cup \overline{P}$,

(d) $T \cup (R \cap P)$,

(e) $\overline{R} \cap (P \cup R \cup T)$.

5 Conditional Probability

A lecture on a topic of public health is held and 300 people attend. They are classified in the following way:

Gender	Doctors	Nurses	Total
Female	90	90	180
Male	100	20	120
Total	190	110	300

If one person is selected at random, find the following probabilities:

(a) P(a doctor is chosen);

(b) P(a female is chosen);

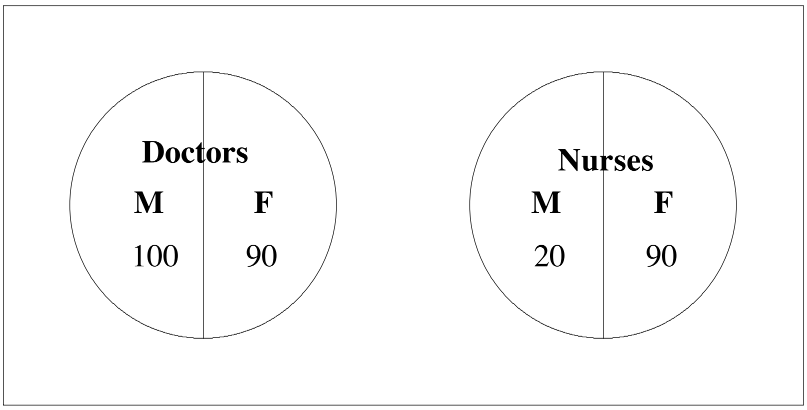
(c) P(a nurse is chosen);

(d) P(a male is chosen);

(e) P(a female nurse is chosen);

(f) P(a male doctor is chosen).

Solution:



(a) The number of doctors is 190 and the total number of people is 300, so P(doctor) $= \frac{190}{300}$

(b) $P(\text{female}) = \frac{180}{300}$

(c) $P(\text{male}) = \frac{120}{300}$

(d) $P(\text{nurse}) = \frac{110}{300}$

(e) There are 90 female nurses, therefore $P(\text{female} \cap \text{nurse}) = \frac{90}{300}$

(f) $P(\text{male doctor}) = P(\text{male} \cap \text{doctor}) = \frac{100}{300}$.

Now suppose you are given the information that a female is chosen and you wish to find the probability that she is a nurse. This is what we call **conditional probability**. We want the probability that the person chosen is a nurse, subject to the condition that we know she is female. The notation used for this is:

$$P(\text{nurse} \mid \text{female})$$

Read this as 'the probability of the person chosen being a nurse, **given** that she is female'. Since there are 180 females and of these 90 are nurses, the required probability is $\frac{90}{180} = \frac{1}{2}$.

We can see that
$$\begin{aligned} P(\text{nurse} \mid \text{female}) &= \frac{90}{180} \\ &= \frac{90/300}{180/300} \\ &= \frac{P(\text{nurse} \cap \text{female})}{P(\text{female})} \end{aligned}$$

Definition

The conditional probability of A given B is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ provided that } P(B) \neq 0.$$

Hence if A and B are any two events with probabilities greater than 0, then

$P(A \cap B) = P(A \mid B).P(B)$

or $P(B \mid A).P(A)$, as this is $P(B \cap A)$ which is equal to $P(A \cap B)$.

Exercise 7

In the example above, find

$P(\text{female} \mid \text{nurse})$, $P(\text{doctor} \mid \text{male})$, $P(\text{male} \mid \text{doctor})$.

Note that the order matters here: $P(A \mid B)$ is not the same as $P(B \mid A)$.

6 Independence

Here is a game of chance. A friend tosses a coin and you bet on the outcome. Suppose she has tossed the coin 3 times and obtained 'heads' all three times. What would you bet on the fourth trial?

You might be inclined to guess 'tails' but would still only have probability $\frac{1}{2}$ of being right.

The chance of getting 'tails' on any one throw is $\frac{1}{2}$. The outcome of one throw is not affected by previous ones - the coin has no memory! (You might, of course, check that the coin does have a tail.)

When the chance of a given outcome remains the same, irrespective of whether or not another event has occurred, the events are said to be **independent**.

Definition:

Two events A and B are said to be independent if and only if $P(A \mid B) = P(A)$, that is, when the conditional probability of A **given** B is the same as the probability of A.

Example

In the problem of nurses and doctors given on page 13, define A to be the event 'a nurse is chosen' and B to be the event 'a female is chosen'.

Are the events A and B independent?

Solution

$$\begin{array}{lcccl} P(A \mid B) & = & \frac{90}{180} & = & \frac{1}{2} \\ P(A) & = & \frac{110}{300} & \neq & P(A \mid B) \end{array}$$

So A and B are not independent.

Note: From the definition of conditional probability we have

$$P(A \cap B) = P(B \mid A).P(A)$$

Now if A and B are independent, then $P(A \mid B) = P(A)$, so $P(A \cap B) = P(A).P(B)$.

When two events are **independent**, the chance that **both** will happen is found by **multiplying** their individual chances.

This gives us a simple way of checking whether or not events are independent:

A and B are independent events if and only if $P(A \cap B) = P(A).P(B)$.

Note: It is possible to define independence in this way without referring to conditional probability.

Example

What is the probability of obtaining 'six' and 'six' on two successive rolls of a die?

Solution

P(obtaining 6 on a roll of a die) $= \frac{1}{6}$.

The two rolls are independent - the die cannot remember what happened first.

So $P(6 \text{ and } 6) = \frac{1}{6}.\frac{1}{6} = \frac{1}{66}$.

Notice that this is equal to the probability of '2' followed by '3', or indeed any ordered sequence of two numbers.

Example

A box contains three white cards and three black cards numbered as follows:

White			Black		
1	2	2	1	1	2

One card is picked out of the box at random. If A is the event 'the card is black' and B is the event 'the card is marked 2', are A and B independent?

Solution

$P(A) = \frac{1}{2}$ since 3 of the cards are black.

$P(B) = \frac{1}{2}$ since 3 of the cards have '2'.

$P(A \cap B) = P(\text{card is black and marked 2}) = \frac{1}{6}$ since only one card satisfies this condition.

Now $\frac{1}{6} \neq \frac{1}{2} . \frac{1}{2}$, so A and B are not independent.

If we know the card is black then the chance of it being a 'two' is changed:- it is now $\frac{1}{3}$.

Hence the outcome of one event **does** affect the outcome of the other, which again shows that A and B are not independent.

Exercise 8

A couple has two children. Let A be the event 'they have one boy and one girl' and B the event 'they have at most one boy'. Are A and B independent?

Exercise 9

Two different missiles are shot simultaneiously at a practice target. If the probability of the first one hitting the target is $\frac{1}{4}$ and of the second one hitting is $\frac{2}{5}$, what is the probability that

(a) both missiles will hit,

(b) at least one will hit?

7 Summary

1. If there are a finite number of equally likely outcomes of an experiment, the probability of an event A is

$$P(A) = \frac{\text{Number of possible outcomes in } A}{\text{Total number of possible outcomes}}$$

2. The probability of an event happening lies between zero and one. If the event cannot happen, its probability is zero and if it is certain to happen, its probability is one.

3. If two events are **complementary**, i.e. they are mutually exclusive (can't happen together) and make up the whole sample space, then their probabilities add up to 1.

4. For two events A and B the probability of A or B (or both) happening is

   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

   In particular if A and B are **mutually exclusive**, $P(A \cup B) = P(A) + P(B)$. That is, the chance that **at least one of them** will happen equals the sum of their probabilities.

5. The **conditional probability** of A given B is $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, provided that $P(B) \neq 0$.

   So $P(A \cap B) = P(A \mid B).P(B)$.

6. A and B are defined to be independent events if $P(A) = P(A \mid B)$. That is, knowing the outcome of one event does not change the probability of the outcome of the other. From (5.) above we see that in this case $P(A \cap B) = P(A).P(B)$, that is the probability of **both** events happening is the **product** of the individual probabilities.

Exercise 10

Can two events A and B, ever be both mutually exclusive and independent?

8 Solutions to Exercises

1. If a raffle is conducted in such a way that each ticket has an equal chance of being drawn then

   $P(\text{winning}) = \frac{1}{325}$.

2. $P(A) = \frac{1}{5}$,

   $P(B) = \frac{2}{5}$.

3. P(at least one bulb flowers)
   $= 1 - P$(no bulbs flower)
   $= 1 - \frac{1}{8}$
   $= \frac{7}{8}$.

4. $S = \{2, 3, 4, ..., 25\}$

   Let Q be the event 'a perfect square is chosen'. Then $Q = \{4, 9, 16, 25\}$.

   Let R be the event 'a prime number is chosen'.

   Then $R = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$.

   (a) $P(Q) = \frac{4}{24} = \frac{1}{6}$
   (b) $P(R) = \frac{9}{24} = \frac{3}{8}$
   (c) $P(Q \cup R) = P(Q) + P(R) = \frac{13}{24}$.
   (since the events Q and R are mutually exclusive, i.e. disjoint)

5.

Classification	Women	Men	Total
Computer Science	7	12	19
No Computer Science	7	4	11
Total	14	16	30

   Let W be the event 'the chosen person is a woman'.

   Let C be the event 'the chosen person takes computer science'.

   (a) $P(W) = \frac{14}{30}$
   (b) $P(C) = \frac{19}{30}$
   (c) $P(W \cap C) = \frac{7}{30}$, as from the table the number of women taking computer science is 7.
   (d) $P(W \cup C) = P(W) + P(C) - P(W \cap C) = \frac{14}{30} + \frac{19}{30} - \frac{7}{30} = \frac{26}{30}$.

6. (a) $R \cup T$ is the event 'a Rainbow or a Tiger's Eye is drawn'.
   $P(R \cup T) = P(R) + P(T) = \frac{40}{45} = \frac{8}{9}$, as these are mutually exclusive.
   (b) $P(R \cap T)$ is the event 'a Rainbow and a Tiger's Eye is drawn'.
   $P(R \cap T) = 0$.
   (c) $T \cup \overline{P}$ is the event 'a Tiger's Eye or not a Pearl is drawn'. This is equivalent to the event 'a Tiger's Eye or a Rainbow is drawn'.
   $P(T \cup \overline{P}) = \frac{40}{50} = \frac{8}{9}$.
   (d) $T \cup (R \cap P)$ is the event 'a Tiger's Eye is drawn or a Rainbow and a Pearl is drawn'. Since $(R \cap P) = \{\}$, the empty set,
   $P(T \cup (R \cap P)) = P(T) = \frac{23}{45}$.

(e) $\overline{R} \cap (P \cup R \cup T)$ is the event 'not a Rianbow and either a Pearl, a Rainbow or a Tiger's Eye is drawn'.

Since $P \cup R \cup T$ is the whole sample space,
$\overline{R} \cap (P \cup R \cup T) = \overline{R}$ and $P(\overline{R}) = \frac{28}{45}$.

7. $P(\text{female} \mid \text{nurse}) = \frac{90}{110} = \frac{9}{11}$, since there are 110 nurses and of these 90 are female.

   $P(\text{doctor} \mid \text{male}) = \frac{100}{120} = \frac{5}{6}$, since there are 120 males of whom 100 are doctors.

   $P(\text{male} \mid \text{doctor}) = \frac{100}{190} = \frac{10}{19}$, since there are 190 doctors and of these 100 are male.

8. Sample space = {GG, BG, GB, BB}

   Note that a 'girl followed by a boy' is **not** the same event as 'a boy followed by a girl'.

   $A =$ {BG, GB}, $B =$ {GG, BG, GB}, $A \cap B =$ {BG,GB},

   $P(A \cap B) = \frac{2}{4} = \frac{1}{2}$ and $P(A).P(B) = \frac{2}{4}.\frac{3}{4} = \frac{3}{8}$.

   Since $P(A \cap B) \neq P(A).P(B)$, A and B are not independent.

9. Let 'missile one hits target' be denoted by M_1 and 'missile two hits target' by M_2

   Then $P(M_1) = \frac{1}{4}$ and $P(M_2) = \frac{2}{5}$,

   (a) $P(M_1 \cap M_2) = \frac{1}{4} \times \frac{2}{5}$, since the events are independent
   $= \frac{1}{10}$

   (b) $P(M_1 \cup M_2) = P(M_1) + P(M_2) - P(M_1 \cap M_2)$ since these are not mutually exclusive
   $= \frac{1}{4} + \frac{2}{5} - \frac{1}{10}$
   $= \frac{11}{20}$.

10. If events A and B are mutually exclusive, then $P(A \cap B) = 0$. If events A and B are independent, then $P(A \cap B) = P(A).P(B)$.

    So they can be mutually exclusive **and** independent only if $P(A).P(B) = 0$.

    This can only happen if either $P(A) = 0$ or $P(B) = 0$, that is, either the even A or event B is impossible.