Chapter 3

Combinatorial Analysis

Combinatorial analysis is a branch of mathematics that studies how to count objects. It provides counting methods that are particularly useful in probability theory.

1 Introduction

The aim of combinatorial analysis is to count the dispositions (layouts) that can be formed from the elements of a finite set of objects. An object is characterized by:

- the place it occupies in the disposition;
- the number of times it can appear.

1.1 Notion of repetition

If an element appears more than once in a layout, the layout is said to have repetition; otherwise, the layout is said to have no repetition.

1.2 Notion of order

A layout is said to be ordered when each time an element changes place (or position), the layout changes.

Example 4.1. Consider a set E with three elements $E = \{a, b, c\}$. Choosing two elements from this set can be done in several different ways.
The following table shows all possible cases:

disposition	with repetition	without repetition
with order (ordinate)	aa, ab, ac, ba, bb, bc, ca, cb, cc	ab, ac, ba, bc, ca, cb
out of order (unordered)	aa, ab, ac, bb, bc, cc	ab, ac, bc

1.3 Factorial of an integer n

Let n be a natural integer, the factorial of n, noted $n!$ is:

$$n! = n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1.$$

Conventionally, we have: $0! = 1$ and $1! = 1$.

Example 4.2.

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 = 120. \\ 10! &= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3628800. \end{aligned}$$

2 Arrangements

Definition 4.1. Given a set E of n objects, an arrangement of p of these objects is an **ordered** sequence of p objects taken from these n objects.

There are two types of arrangements: with and without repetition.

2.1 Arrangement without repetition

We call an arrangement without repetition of p objects chosen from n objects any **ordered** layout (disposition) of p objects taken from the n objects **without repetitions**.

The number of arrangements without repetition, noted A_n^p, is as follows:

$$A_n^p = n \times (n-1) \times (n-2)..... \times (n-p+1) = \frac{n!}{(n-p)!},$$

where $1 \leq p \leq n$.

In an arrangement without repetition, the p objects in the list are all distinct. This corresponds to a draw without replacement and with order.

Example 4.3. How many three-letter words containing no more than one letter can be formed using the letters of the alphabet?

$$A_{26}^3 = \frac{26!}{(26-3)!} = 26 \times 25 \times 24 = 15600 \text{ mots}.$$

2.2 Arrangement with repetition

We call an arrangement with repetition of p objects chosen from n objects any **ordered** layout (disposition) of p objects taken from the n objects **with repetitions**.

The number of arrangements with repetition, noted n^p, is as follows:

$$n^p = \underbrace{n \times n \times n ... \times n}_{p \text{ times}}$$

avec $1 \leq p \leq n$.

In a non-repetition arrangement, the p objects in the list are not necessarily all distinct. This corresponds to a draw **with replacement** and **with order**.

Example 4.4. How many two-letter words can be made with the letters of the alphabet?

$$26^2 = 26 \times 26 = 676 \text{ mots}.$$

3 Permutations

Definition 4.2. Let E a set of n objects. We call permutation of n distinct objects **any ordered sequence** of n objects or **any arrangement** n to n of these objects.

3.1 Permutation without repetition

This is the special case of the arrangement without repetition of p objects among n objects, when $p = n$.

The number of permutations of n objects is:

$$P_n = n!.$$

Remark 4.1.

$$P_n = A_n^n == \frac{n!}{(n-n)!} = n!.$$

Example 4.5. The number of ways to seat eight diners (guests) around a table is:

$$P_8 = 8! = 40320.$$

3.2 Permutation with repetition

In the case where there are k identical objects among the n objects, then

$$P_n = \frac{n!}{k!}.$$

Example 4.6. The number of possible words (with or without meaning) that can be formed by permuting the 8 letters of the word "Quantity" is

$$P_8 = \frac{8!}{2!} = 20160 \text{ words, we have 2 } \mathbf{t} \text{ in "Quan}\mathbf{t}\text{i}\mathbf{t}\text{y".}$$

Considering the word "Swimming", the number of possible words is

$$P_8 = \frac{8!}{2!2!} = 10080 \text{ words, because we have the } \mathbf{i} \text{ 2 times and the } \mathbf{m} \text{ 2 times.}$$

4 Combinations

4.1 Combination without repetitions (without discounts)

Definition 4.3. Given a set E of n objects. We call combinations of p objects **any set** of p objects taken from the n objects without replacement (without discount).

The number of combinations of p objects among n and without replacement, is:

$$C_n^p = \frac{n!}{(n-p)!p!},$$

where $1 \leq p \leq n$.

Remark 4.2.

$$C_n^p = \frac{A_n^p}{p!} = \frac{n!}{(n-p)!p!}.$$

Example 4.7. The random drawing of 5 cards from a deck of 32 cards (poker hand) is a combination with $p = 5$ and $n = 32$. The number of possible drawings is

$$C_{32}^5 = \frac{32!}{(32-5)!5!} = 409696 \text{ possibilities.}$$

Example 4.8. Forming a delegation of 2 students from a group of 20 is a combination with $p = 2$ and $n = 20$. The number of possible delegations is

$$C_{20}^2 = \frac{20!}{(20-2)!2!} = 190 \text{ possibilities.}$$

4.2 Combination with repetitions (with discounts)

The number of combinations of p objects among n and with replacement (with discount), is:

$$C_{n+p-1}^{p} = \frac{(n+p-1)!}{p!(n-1)!}.$$

Example 4.9. Let's make up 3-letter words from a 5-letter alphabet with discount.
The number of words is

$$C_{5+3-1}^{3} = C_7^3 = 35.$$

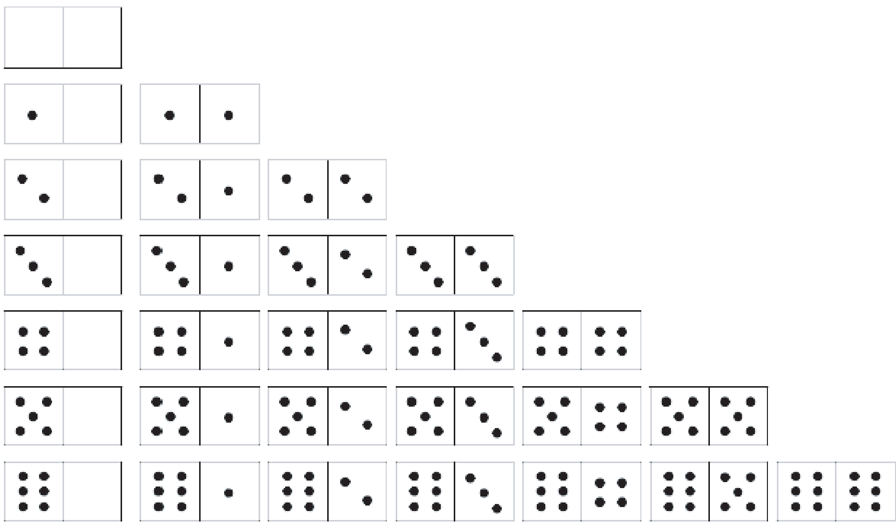
There are 3 possible cases:

- C_5^3 number of words of 3 different letters and without order;
- $2C_5^2$ number of words with 2 different letters and one redundant letter;
- C_5^1 number of words with 3 identical letters;

in total, we have $C_5^3 + 2C_5^2 + C_5^1 = C_7^3 = 35$ words.

Example 4.10. We want to know the number of dominoes that can be formed with the numbers: 0, 1, 2, 3, 4, 5 and 6. Each half of the domino can take 7 values (corresponding to the number of points present: 0 to 6).

First method :



Simply count the number of dominoes per row (or column) and add up all the numbers (see figure). We get:

$$N = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 \text{ dominoes}.$$

Second method :

To simplify the counting, we can distinguish double dominoes (both halves have the same number of points) from the others. We get 7 double dominoes that we can write down : $(0,0)$, $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$ and $(6,6)$.

As for the other dominoes, you simply have to draw **without discounts** two values (2 elements) of the set $\{0,1,2,3,4,5,6\}$, the order being unimportant. We must therefore determine the number N_2 of **combinations without repetitions** of two elements among 7:

$$N_2 = C_7^2 = \frac{7!}{(7-2)!2!} = \frac{7\times 6}{2} = 21.$$

Hence the number of dominoes is $N = N_2 + 7 = 21 + 7 = 28$.

Third method :

This time, we draw two values **with discount** from the set $\{0,1,2,3,4,5,6\}$, the order being unimportant. We must therefore determine the number N of **combinations with repetitions** of two elements among 7:

$$N = C_{7+2-1}^2 = \frac{8!}{(8-2)!2!} = \frac{8\times 7}{2} = 28 \text{ dominoes.}$$

4.3 Properties of combinations and Newton's binomial

We have

$$C_n^0 = C_n^n = \frac{n!}{n!} = 1.$$

$$\forall n \geq 1,\ C_n^1 = C_n^{n-1} = \frac{n!}{(n-1)!} = n.$$

$$\forall n \geq 2,\ C_n^2 = C_n^{n-2} = \frac{n(n-1)}{2}.$$

Therefore, for $0 \leq p \leq n$, we have $C_n^p = C_n^{n-p}$.

The compound combination also called Pascal's formula: for $0 \leq p \leq n-1$, we have

$$C_{n-1}^{p-1} + C_{n-1}^p = C_n^p.$$

Theorem 4.1. ***(The Binomial Theorem)***

Newton's binomial theorem gives the general expression for the development of $(a+b)^n$.

$$\begin{aligned}(a+b)^n &= \sum_{p=0}^{n} C_n^p a^{n-p} b^p \\ &= C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + \ldots + C_n^n a^0 b^n \\ &= a^n + n a^{n-1} b + \ldots + b^n.\end{aligned}$$

Example 4.11. For $n = 4$, we have

$$\begin{aligned}(a+b)^4 &= \sum_{p=0}^{4} C_4^p a^{4-p} b^p \\ &= C_4^0 a^4 b^0 + C_4^1 a^3 b^1 + \ldots + C_4^4 a^0 b^4 \\ &= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4.\end{aligned}$$

And for $n = 5$,

$$(a+b)^5 = a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5.$$

Remark 4.3. When $a = b = 1$, we have $(1+1)^n = 2^n$ and $2^n = \sum_{p=0}^{n} C_n^p$. Then

$$2^n = C_n^0 + C_n^1 + C_n^2 + \ldots + C_n^n.$$

5 Corrected exercises

5.1 Exercises

Exercise 1: A keypad allows you to enter a building code using a letter followed by a 3-digit number, whether or not they are distinct.

1. How many different codes can be formed?
2. How many codes are there without the digit 1?
3. How many codes are there containing at least one digit 1?
4. How many codes are there containing distinct digits?

Exercise 2: How many distinct words, with or without meaning, can be formed with all the letters of each of the words:

1. maths;
2. statistics.

Exercise 3: We take 3 bulbs out of 15 simultaneously (at the same time), 5 of which are defective.

1. In how many different ways can this draw be made?
2. In how many different ways can at least one defective bulb be obtained?

Exercise 4: A coat rack has 5 coat hangers in a row. How many distinct layouts (dispositions) and without stacking coats can be hung on it:

1. three coats?
2. five coats?
3. six coats?

Exercise 5: Four mathematicians and two physicists sit on a six-seater bench.

1. How many layouts are possible?
2. Same question if mathematicians are on one side and physicists on the other?
3. Same question if each physicist sits between two mathematicians?
4. Same question if the physicists want to stay next to each other?

Exercise 6: An urn contains one white ball, three black balls and four red balls. In the following questions, three balls are drawn at random.

1. In how many ways can three balls be drawn successively and without replacement from this urn?
2. In how many ways can three balls be drawn simultaneously and without replacement from this urn?
3. In how many ways can three balls of different colors be drawn simultaneously?
4. In how many ways can two red balls and one white ball be drawn successively and with replacement?

5.2 Corrections

Exercise 1: The code of a building consists of a letter followed by a number of 3 distinct or non-distinct digits.

The code ↠ | Letter | digit | digit | digit |

A letter among the 26 letters of the alphabet.
Three digits out of ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

1. The number of codes that can be formed:

$$\underbrace{\text{order is important and with repetitions}}_{\text{arrangement with repetitions} \quad (n^p)}$$

$$N_1 = 26 \times n^p = 26 \times 10^3 = 26000 \text{ codes.}$$

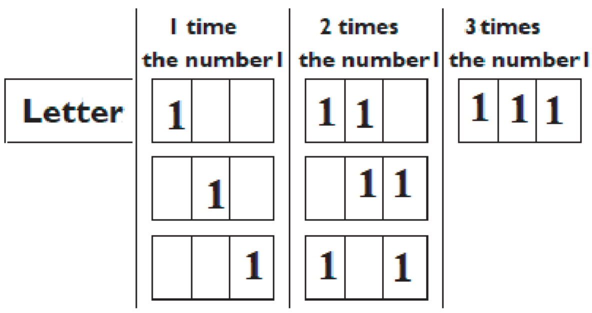
2. The number of codes without the number 1:
The code is made up of three digits among the 9 digits: 0, 2, 3, 4, 5, 6, 7, 8 and 9.

$$N_2 = 26 \times (n-1)^p = 26 \times 9^3 = 18954 \text{ codes.}$$

Remark: We will talk about an arrangement without repetitions ($A_n^p = \frac{n!}{(n-p)!}$) when the numbers are distinct.
The number of codes without the digit 1 and with distinct (different) digits is:

$$N_2' = 26 \times A_9^3 = 26 \times \frac{9!}{(9-3)!} = 26 \times 9 \times 8 \times 7 = 13104 \text{ codes.}$$

3. The number of codes containing at least one number 1:
First Method: $N_3 = N_1 - N_2 = 26000 - 18954 = 7046$ codes.
Second Method:



$$N_3 = 26\Big(3(1^1 \times 9^2) + 3(1^2 \times 9^1) + 1^3\Big) = 7046 \text{ codes.}$$

4. The number of codes containing distinct digits is

$$N_4 = 26 \times A_{10}^3 = 26 \times \frac{10!}{(10-3)!} = 26 \times 10 \times 9 \times 8 = 18720 \text{ codes.}$$

Exercise 2: The number of distinct words, with or without meaning, that can be formed with all the letters of each of the words: maths and statistics.

1. maths (Permutation without repetitions)
   $n = 5$

$$N_{maths} = n! = 5! = 120 \text{ words.}$$

2. statistics (Permutation with repetitions)
   $n = 10$, $n_i = 2$, $n_s = 3$ et $n_t = 3$ (2i, 3s and 3t).

$$N_{statistics} = \frac{n!}{n_i! n_s! n_t!} = \frac{10!}{2!3!3!} = 50400 \text{ words.}$$

Exercise 3: We simultaneously draw 3 bulbs out of 15, 5 of which are defective.

$$\underbrace{\text{the order is not important and without discounts}}_{\text{Combinations without repetitions} \quad (C_n^p = \frac{n!}{(n-p)!p!})}$$

1. The number of possible draws is:

$$N_1 = C_{15}^3 = \frac{15!}{(15-3)!3!} = 455.$$

2. Get at least one defective bulb:
   Three possibilities: **one** defective bulb, **two** defective bulbs or **three** defective bulbs.

$$\begin{aligned} N_2 &= (C_5^1 \times C_{10}^2) + (C_5^2 \times C_{10}^1) + (C_5^3 \times C_{10}^0) \\ &= (\frac{5!}{(5-1)!1!} \times \frac{10!}{(10-2)!2!}) + (\frac{5!}{(5-2)!2!} \times \frac{10!}{(10-1)!1!}) + (\frac{5!}{(5-3)!3!} \times \frac{10!}{(10-0)!0!}) \\ &= (5 \times 45) + (10 \times 10) + (10 \times 1) = 335. \end{aligned}$$

Exercise 4: A coat rack consists of 5 hangers lined up. We want to hang each coat on a hanger without stacking the coats.

1. **Number of possible layouts for three coats:**
   Arrangements without repetitions

$$N_1 = A_5^3 = \frac{5!}{(5-3)!} = 5 \times 4 \times 3 = 60.$$

2. **Number of possible layouts for five coats:**
   Permutations without repetitions
   $$N_2 = 5! = 120.$$
3. **Number of possible layouts for six coats:**
   impossible d'accrocher six manteaux sur le portemanteau (5 porteurs) sans empiler les manteaux.

Exercise 5: Four mathematicians and two physicists sit on a bench with six seats.
Each mathematician is marked by **M** and each physicist by **P**.
The order is important and without repetition $\Rrightarrow$ an arrangement without repetition.

1. Number of possible layouts:
   $$n = 4{+}2 = 6 \text{ and the 6-seater bench} \Rrightarrow \begin{cases} n = p = 6, \\ \text{permutations without repetitions.} \end{cases}$$
   $N_1 = n! = 6! = 720.$
2. Mathematicians are on one side and physicists on the other:
   Possible cases: **MMMMPP** and **PPMMMM**.
   Permutations without repetitions for the 2 physicists and permutations without repetitions for the 4 mathematicians, separately.
   The number of dispositions is: $N_2 = 2(4!2!) = 96$.
3. Every physicist sits between two mathematicians:
   Possible cases: **MPMPMM**; **MPMMPM** and **MMPMPM**.
   The number of dispositions is: $N_3 = 3(4!2!) = 144$.
4. Physicists want to stand next to each other:
   Possible cases:
   PPMMMM; **MPPMMM**; **MMPPMM**; **MMMPPM**; **MMMMPP**.
   The number of dispositions is: $N_4 = 5(4!2!) = 240$.

Exercise 6: The urn contains 8 balls: 1 white ball + 3 black balls + 4 red balls.

1. The number of successive draws of three balls from this urn without replacements (discounts):
   $$N_1 = A_8^3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \times 7 \times 6 = 336 \text{ ways.}$$

2. The number of simultaneous draws of three balls without replacements:

$$N_2 = C_8^3 = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!} = \frac{8 \times 7 \times 6}{6} = 56 \text{ ways.}$$

3. The number of simultaneous draws of three balls without replacements of different colors:
   Balls of different colors means 1 white ball + 1 black ball + 1 red ball.

$$N_3 = C_1^1 C_3^1 C_4^1 = 1 \times 3 \times 4 = 12 \text{ ways.}$$

4. The number of successive draws with the replacement of two red balls and one white ball:
   Let us denote the red balls by R and the white balls by W, then the possible layouts for 2 red balls and 1 white ball are: WRR ou RWR ou RRW.

$$N_4 = 3(1^1 \times 4^2) = 48 \text{ ways.}$$

References:

Hacene Gharout,
« Cours de Mathématiques (cours, exercices de Maths et de statistiques) »,
ISBN : 978-9969-9753-3-8,
Editeur : Maison des universitaires pour l'Edition, l'imprimerie et la diffusion,
Année : 2024.