Model Solution-Tutorial exam No. 1

Exercise 1 : 4 Pts

1. Since $\dim(\mathbb{R}_2[X])^* = 3 = \text{Card}\{\psi_1, \psi_2, \psi_3\}$, it suffices to prove that the family is linearly independent. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and assume

$$\alpha_1\psi_1 + \alpha_2\psi_2 + \alpha_3\psi_3 = 0.$$

That is,

$$\alpha_1 P(1) + \alpha_2 P'(0) + \alpha_3 \int_{-1}^{0} P(t)\,dt = 0, \quad \forall P \in \mathbb{R}_2[X].$$

Taking successively $P(X) = 1$, $P(X) = X$, and $P(X) = X^2$, we obtain

$$\begin{cases} \alpha_1 + \alpha_3 = 0, \\ \alpha_1 + \alpha_2 - \frac{1}{2}\alpha_3 = 0, \\ \alpha_1 + \frac{1}{3}\alpha_3 = 0. \end{cases}$$

Hence $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and the family is linearly independent. Therefore, $\{\psi_1, \psi_2, \psi_3\}$ is a basis of $(\mathbb{R}_2[X])^*$.

2. Let $\mathcal{B} = \{Q_1, Q_2, Q_3\}$ be the predual basis, where

$$Q_j(X) = a_j X^2 + b_j X + c_j, \quad j = 1, 2, 3.$$

The conditions $\psi_i(Q_j) = \delta_{ij}$ yield the following systems.

For Q_1 :

$$\begin{cases} a_1 + b_1 + c_1 = 1, \\ b_1 = 0, \\ \frac{a_1}{3} - \frac{b_1}{2} + c_1 = 0, \end{cases} \Rightarrow a_1 = \frac{3}{2},\ b_1 = 0,\ c_1 = -\frac{1}{2}.$$

For Q_2 :

$$\begin{cases} a_2 + b_2 + c_2 = 0, \\ b_2 = 1, \\ \frac{a_2}{3} - \frac{b_2}{2} + c_2 = 0, \end{cases} \Rightarrow a_2 = -\frac{9}{4},\ b_2 = 1,\ c_2 = \frac{5}{4}.$$

For Q_3 :

$$\begin{cases} a_3 + b_3 + c_3 = 0, \\ b_3 = 0, \\ \frac{a_3}{3} - \frac{b_3}{2} + c_3 = 1, \end{cases} \Rightarrow a_3 = -\frac{3}{2},\ b_3 = 0,\ c_3 = \frac{3}{2}.$$

Thus,

$$\mathcal{B} = \left\{ \frac{3}{2}X^2 - \frac{1}{2},\ -\frac{9}{4}X^2 + X + \frac{5}{4},\ -\frac{3}{2}X^2 + \frac{3}{2} \right\}.$$

Exercise 1 : 2 Pts

By the Rank Theorem, we have

$$\dim \ker f + \dim \operatorname{Im} f = \dim E.$$

Since f is nonzero, $\operatorname{Im} f \neq \{0_K\}$ and $\operatorname{Im} f$ is a subspace of K. Thus

$$0 < \dim \operatorname{Im} f \leq \dim K = 1,$$

which implies $\dim \operatorname{Im} f = 1$, and therefore

$$\dim \ker f = n - 1.$$