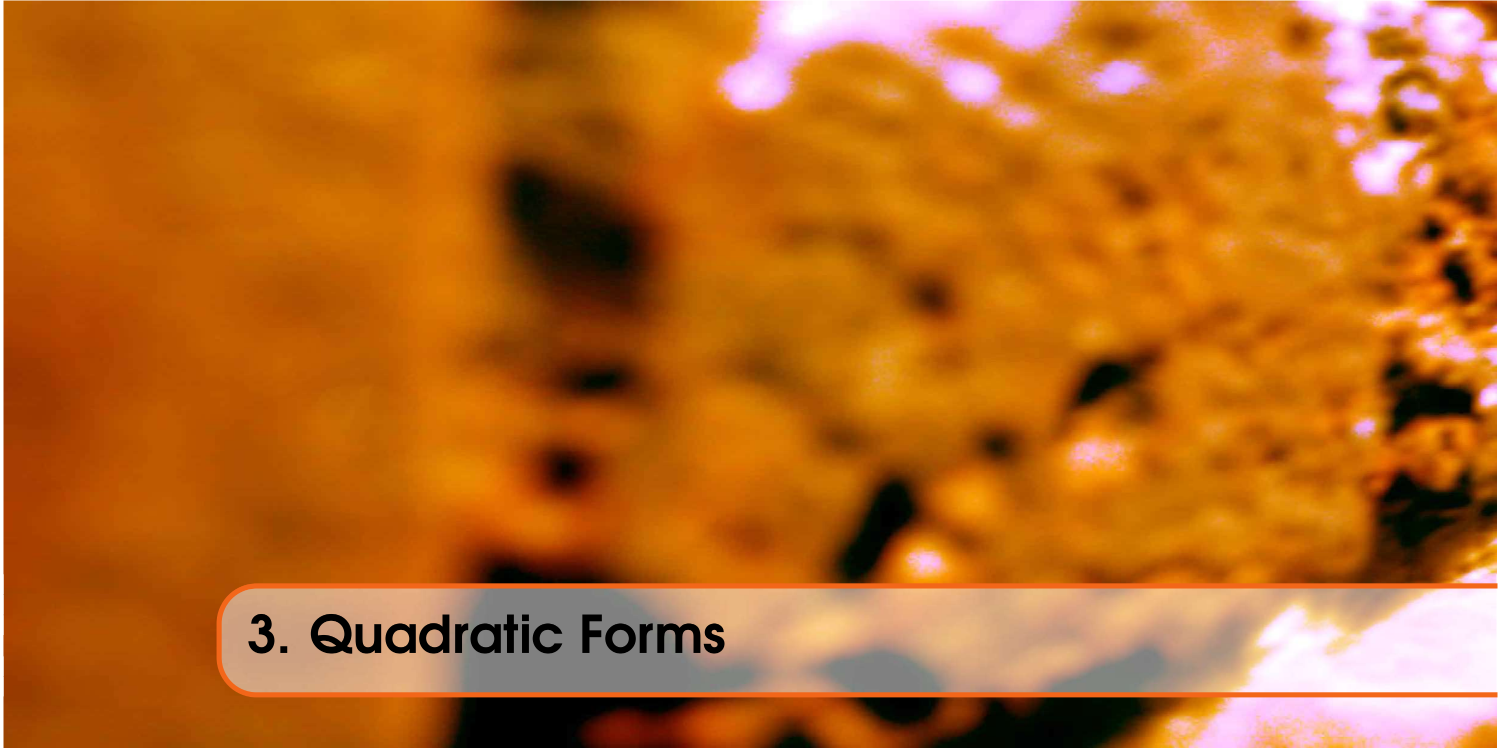

3. Quadratic Forms

3.1 Quadratic Forms

Definition 3.1.1 Let $f \in \mathscr{L}(E \times E, \mathbb{K})$. The mapping

$$\begin{aligned} q: \quad E &\longrightarrow \mathbb{K} \\ x &\longmapsto q(x) = f(x,x) \end{aligned}$$

is called the **quadratic form associated with** f.

The mapping f is called the **polar form associated with** q.

■ **Example 3.1**

1. Let the symmetric bilinear form

$$\begin{aligned} \varphi: \quad \mathscr{C}([0,1],\mathbb{R}) \times \mathscr{C}([0,1],\mathbb{R}) &\longrightarrow \mathbb{R} \\ (f,g) &\longmapsto \int_0^1 f(t)g(t)\,dt \end{aligned}$$

The quadratic form associated with φ is

$$\begin{aligned} q_\varphi: \quad \mathscr{C}([0,1],\mathbb{R}) &\longrightarrow \mathbb{R} \\ f &\longmapsto q_\varphi(f) = \varphi(f,f) = \int_0^1 f(t)^2\,dt \end{aligned}$$

2. Let the symmetric bilinear form

$$\begin{aligned} f: \quad \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ \big((x_1,x_2),(y_1,y_2)\big) &\longmapsto x_1y_1 - x_2y_2 \end{aligned}$$

The quadratic form associated with f is

$$\begin{aligned} q_f: \quad \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x_1,x_2) &\longmapsto q_f(x_1,x_2) = x_1^2 - x_2^2 \end{aligned}$$

■

3.1.1 Properties of Quadratic Forms

Proposition 3.1.1 Let q be a quadratic form on E, a vector space over $\mathbb{K}$, and let f be its polar form.

i) $q(\alpha x) = \alpha^2 q(x), \quad \forall x \in E, \forall \alpha \in \mathbb{K}$.

ii) $q(x+y) = q(x) + q(y) + 2f(x,y), \quad \forall x, y \in E \quad$ (Polarization identity).
iii) $q(x+y) + q(x-y) = 2(q(x) + q(y)), \quad \forall x, y \in E \quad$ (Parallelogram identity).

Proof. **i)** Let $x \in E$ and $\alpha \in \mathbb{K}$. Then

$$q(\alpha x) = f(\alpha x, \alpha x) = \alpha\alpha f(x,x) = \alpha^2 q(x).$$

ii) Let $x, y \in E$. Then

$$\begin{aligned} q(x+y) &= f(x+y, x+y) \\ &= f(x,x) + f(x,y) + f(y,x) + f(y,y) \\ &= q(x) + 2f(x,y) + q(y), \end{aligned}$$

since f is symmetric.

iii) From (ii) we have

$$q(x+y) = q(x) + q(y) + 2f(x,y),$$

$$q(x-y) = q(x) + q(y) - 2f(x,y).$$

Adding these two equalities yields

$$q(x+y) + q(x-y) = 2(q(x) + q(y)).$$

■

R The polarization identity allows one to recover f from q:

$$f(x,y) = \frac{1}{2}\left[q(x+y) - q(x) - q(y)\right].$$

■ **Example 3.2** Let the quadratic form

$$\begin{aligned} q : \quad & \mathbb{R}^2 \longrightarrow \mathbb{R} \\ & (x_1, x_2) \longmapsto q(x_1, x_2) = x_1^2 + 2x_2^2 - 3x_1x_2. \end{aligned}$$

The polar form associated with q is

$$\begin{aligned} f : \quad & \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R} \\ & ((x_1, x_2), (y_1, y_2)) \longmapsto f((x_1, x_2), (y_1, y_2)), \end{aligned}$$

where

$$\begin{aligned} f((x_1,x_2),(y_1,y_2)) &= \frac{1}{2}\left[q((x_1,x_2)+(y_1,y_2)) - q((x_1,x_2)) - q((y_1,y_2))\right] \\ &= \frac{1}{2}\left[q(x_1+y_1, x_2+y_2) - q(x_1,x_2) - q(y_1,y_2)\right] \\ &= x_1y_1 + 2x_2y_2 - \frac{3}{2}x_1y_2 - \frac{3}{2}x_2y_1. \end{aligned}$$

■

3.1.2 Isotropic Cone

Definition 3.1.2 Let q be a quadratic form on E, a vector space over $\mathbb{K}$. A vector $x \in E$ is called an **isotropic vector** if $q(x) = 0$.

The set of all isotropic vectors of q, denoted by $C(q)$, is called the **isotropic cone**, that is,

$$C(q) = \{x \in E \; ; \; q(x) = 0\}.$$

R

1. Since $q(0) = 0$, the zero vector is always isotropic.
2. In general, the isotropic cone is not a vector subspace of E.

■ **Example 3.3** Let the quadratic form

$$\begin{array}{rcl} q : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x_1, x_2) & \longmapsto & q(x_1, x_2) = x_1^2 - x_2^2. \end{array}$$

Then

$$\begin{array}{rcl} C(q) & = & \{(x_1, x_2) \; ; \; q(x_1, x_2) = 0\} \\ & = & \left\{(x_1, x_2) \; ; \; x_1^2 - x_2^2 = 0\right\} \\ & = & \{(x_1, \pm x_1) \; ; \; x_1 \in \mathbb{R}\}. \end{array}$$

■

3.1.3 Kernel of a Quadratic Form

Definition 3.1.3 The **kernel** of a quadratic form, denoted by $N(q)$, is the kernel of its polar form. That is,

$$N(q) = \mathrm{Ann}(f) = \{x \in E \; ; \; f(x, y) = 0 \; \forall y \in E\}.$$

3.1.4 Matrix Associated with a Quadratic Form

Definition 3.1.4 Let q be a quadratic form on E, a finite-dimensional vector space over $\mathbb{K}$, and let B be a basis of E. The matrix associated with q with respect to the basis B, denoted by $\mathscr{M}_B(q)$, is the matrix associated with the polar form f of q with respect to B, that is,

$$\mathscr{M}_B(q) = \mathscr{M}_B(f).$$

R The matrix associated with a quadratic form is symmetric, that is,

$$\mathscr{M}_B(q) = {}^t\mathscr{M}_B(q).$$

Definition 3.1.5 Let q be a quadratic form on $\mathbb{R}^n$ and let $B = \{e_1, e_2, \ldots, e_n\}$ be the canonical basis. The matrix associated with q with respect to B is

$$M_B(q) = \frac{1}{2} \begin{pmatrix} \frac{\partial q}{\partial x_1}(e_1) & \frac{\partial q}{\partial x_2}(e_1) & \cdots & \frac{\partial q}{\partial x_n}(e_1) \\ \frac{\partial q}{\partial x_1}(e_2) & \frac{\partial q}{\partial x_2}(e_2) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q}{\partial x_1}(e_n) & \frac{\partial q}{\partial x_2}(e_n) & \cdots & \frac{\partial q}{\partial x_n}(e_n) \end{pmatrix}.$$

■ **Example 3.4** Let the quadratic form

$$\begin{array}{rcl} q:\mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ (x_1,x_2,x_3) & \longmapsto & x_1^2+3x_2^2+3x_3^2+2x_1x_2-4x_1x_3. \end{array}$$

We compute

$$\begin{aligned} \frac{\partial q}{\partial x_1} &= 2x_1+2x_2-4x_3, \\ \frac{\partial q}{\partial x_2} &= 2x_1+6x_2, \\ \frac{\partial q}{\partial x_3} &= -4x_1+6x_3. \end{aligned}$$

Therefore, the matrix associated with q with respect to the canonical basis of $\mathbb{R}^3$ is

$$M_B(q)=\frac{1}{2}\begin{pmatrix} 2 & 2 & -4 \\ 2 & 6 & 0 \\ -4 & 0 & 6 \end{pmatrix}=\begin{pmatrix} 1 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 3 \end{pmatrix}.$$

■

3.1.5 Quadratic Form Associated with a Square Matrix

Proposition 3.1.2 Let $M=(a_{ij})$ be a symmetric matrix associated with the quadratic form q with respect to the canonical basis $\{e_1,e_2,\dots,e_n\}$ of E. Then

$$\forall x=\sum_{i=1}^{n}x_ie_i,\quad q(x)=\sum_{i=1}^{n}a_{ii}x_i^2+2\sum_{1\le i<j\le n}a_{ij}x_ix_j.$$

Proof. Let $M=(a_{ij})$ be a symmetric matrix ($a_{ij}=a_{ji}$ or ${}^tM=M$):

$$M=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Let f be the polar form associated with q. Then

$$\begin{aligned} q(x) &= f(x,x) \\ &= {}^txMx \\ &= (x_1,x_2,\dots,x_n)M\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \end{aligned}$$

Thus,

$$q(x)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j.$$

Since M is symmetric ($a_{ij}=a_{ji}$), we separate diagonal and off-diagonal terms:

$$q(x)=\sum_{i=1}^{n}a_{ii}x_i^2+2\sum_{1\le i<j\le n}a_{ij}x_ix_j.$$

■

R The diagonal entries of M are the coefficients of the terms x_i^2.

Example 3.5 Let the symmetric matrix

$$M = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 2 & 0 \\ -2 & 0 & -1 \end{pmatrix}.$$

The quadratic form q associated with M is

$$\begin{array}{rcl} q : \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ (x_1, x_2, x_3) & \longmapsto & q(x_1, x_2, x_3), \end{array}$$

where

$$\begin{aligned} q(x_1,x_2,x_3) &= \sum_{i=1}^{3} a_{ii}x_i^2 + 2 \sum_{1\le i<j\le 3} a_{ij}x_ix_j \\ &= x_1^2 + 2x_2^2 - x_3^2 + 2(3x_1x_2 - 2x_1x_3) \\ &= x_1^2 + 2x_2^2 - x_3^2 + 6x_1x_2 - 4x_1x_3. \end{aligned}$$

■

3.1.6 Rank of a Quadratic Form

Definition 3.1.6 The **rank** of a quadratic form q, denoted by $\text{rank}(q)$, is the rank of its associated matrix with respect to any basis of E.

Reduction of Quadratic Forms via Gauss Method

Gauss Reduction Process

In such a basis, the matrix associated with q relative to $\mathscr{B}$ is diagonal. If this matrix is denoted by

$$D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \quad (\text{with } \lambda_1, \lambda_2, \ldots, \lambda_n \in K),$$

then for every $x = x_1u_1 + x_2u_2 + \cdots + x_nu_n \in E$, we have

$$q(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2.$$

Such a representation of q (i.e., a linear combination of squares of K-linearly independent linear forms) is called a *reduction* (or a reduced form) of q.

Starting from the expression of q relative to an arbitrary basis of E, we shall establish in what follows an efficient algorithm (due to Gauss) that allows one to obtain (in a finite number of steps) a reduced form of q.

Let $\mathscr{B} = (e_1, e_2, \ldots, e_n)$ be a basis of E. According to Proposition 3.2, the expression of q relative to $\mathscr{B}$ can be written (for every $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \in E$, with $x_1, x_2, \ldots, x_n \in K$) as

$$q(x) = \sum_{1\le i,j\le n} a_{ij}x_ix_j,$$

where the a_{ij} are scalars (i.e., elements of K).

The Gauss algorithm for reducing q is a recursive procedure. Its main purpose is to show how to write $q(x)$ as a linear combination of the square of a linear form and a homogeneous polynomial of degree two in $(n-1)$ other linear forms.

The rest is clear: it suffices to repeat this process as many times as necessary until $q(x)$ is expressed as a linear combination of squares of linear forms.

The algorithm guarantees the linear independence of the linear forms that appear at each step; in particular, it ensures the linear independence of the linear forms appearing in the final reduction. These linear forms may therefore be regarded as coordinates relative to a new basis of E. This will precisely be the orthogonal basis corresponding to the reduced form of q.

4.1.1 Description of Gauss's Algorithm

- If $n=1$, then the quadratic form q is already reduced and there is nothing to do.
- Assume from now on that $n>2$. We distinguish the following two cases:
  First case: (if the a_{ii} are not all zero).
  Up to permuting[1] the coordinates $x_1, x_2, \ldots, x_n$, one may assume that $a_{11} \neq 0$. We then write $q(x)$ (for $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \in E$) as a polynomial of degree two in x_1 (with coefficients in $K[x_2, x_3, \ldots, x_n]$), and then put it into canonical form. More precisely, for every $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \in E$, we have:

$$q(x) = a_{11}x_1^2 + \sum_{2\leq j\leq n} a_{1j}x_1x_j + \sum_{2\leq i,j\leq n} a_{ij}x_ix_j$$

$$= a_{11}x_1^2 + \left(\sum_{2\leq j\leq n} a_{1j}x_j\right)x_1 + \sum_{2\leq i,j\leq n} a_{ij}x_ix_j$$

$$= a_{11}\left(x_1^2 + \frac{1}{a_{11}}\sum_{2\leq j\leq n} a_{1j}x_jx_1\right) + \sum_{2\leq i,j\leq n} a_{ij}x_ix_j$$

$$= a_{11}\left[\left(x_1 + \frac{1}{2a_{11}}\sum_{2\leq j\leq n} a_{1j}x_j\right)^2 - \left(\frac{1}{2a_{11}}\sum_{2\leq j\leq n} a_{1j}x_j\right)^2\right] + \sum_{2\leq i,j\leq n} a_{ij}x_ix_j.$$

  Setting

$$L_1(x_1, x_2, \ldots, x_n) = x_1 + \frac{1}{2a_{11}}\sum_{2\leq j\leq n} a_{1j}x_j,$$

  (which is the expression of a linear form on E), we obtain

$$q(x) = a_{11}L_1(x_1, x_2, \ldots, x_n)^2 + \sum_{2\leq i,j\leq n} b_{ij}x_ix_j,$$

  for certain $b_{ij} \in K$ (expressed in terms of the a_{ij}).

[1] A permutation of the coordinates corresponds to a permutation of the vectors of the basis of E under consideration.