Chapter 4

Systems of Linear Equations

Linear equations form the foundation of algebra and have wide applications across mathematics, science, and engineering. This chapter explores systems of linear equations - collections of one or more linear equations involving the same variables.

We will examine:

- Fundamental definitions and properties of linear systems
- Matrix representations of linear equations
- Two powerful solution methods: Cramer's Rule and Gaussian Elimination
- Classification of systems based on their solutions

The study of linear systems is essential for understanding more advanced mathematical concepts and has practical applications in fields ranging from computer graphics to economic modeling. We begin with precise definitions that will allow us to systematically analyze and solve these systems.

4.1 Definitions and Properties

Definition 4.1: Linear Equation

A linear equation in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers, usually known in advance.

Example 4.1: Linear and Non-linear Equations

1. The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2\sqrt{6} - x_1 + x_3$$

are both linear because they can be rearranged algebraically as in equation (1):

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}.$$

2. The equations

$$4x_1 - 5x_2 = x_1 x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

are not linear because of the presence of $x_1 x_2$ in the first equation and $\sqrt{x_1}$ in the second.

Definition 4.2: System of Linear Equations

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables - say, $x_1, \ldots, x_n$. A system of n linear equations in p unknowns with coefficients in $\mathbb{R}$ is:

$$(S)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p = b_2, \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{np}x_p = b_n. \end{cases}$$

Example 4.2: System of Linear Equations

An example is:

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8, \\ x_1 - 4x_3 = 7. \end{cases} \tag{4.1}$$

Definition 4.3: Solution of a Linear System

1. A solution of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

   For instance, $(5, 6.5, 3)$ is a solution of system (4.1) because, when these values are substituted in (2) for x_1, x_2, x_3 respectively, the equations simplify to $8 = 8$ and $7 = 7$.

2. The set of all possible solutions is called the solution set of the linear system.

3. Two linear systems are called equivalent if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Example 4.3: Solution Examples

1. Consider:
$$\begin{cases} 5x - 2y = -7, \\ -2x + y = 2 \end{cases} \Rightarrow (x, y) = (-3, -4).$$

2.
$$\begin{cases} 7x - y = -2, \\ x - y = 4 \end{cases} \Rightarrow (x, y) = (-1, -5).$$

Definition 4.4: Matrix Notation

The essential information of a linear system can be recorded compactly in rectangular arrays called matrices. Given the system of m linear equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

we define the following matrix representations:

- **The coefficient matrix** $(m \times n)$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

- **The constant vector** $(m \times 1)$:

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

- **The variable vector** $(n \times 1)$:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

- **The augmented matrix** $(m \times (n+1))$:

$$[A|B] = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

The system can be expressed in two equivalent forms:

- **Matrix equation form**: $AX = B$,
- **Augmented matrix form**: $[A|B]$.

4.2 Solving a Linear System

This section and the next describe an algorithm, or a systematic procedure, for solving linear systems.

Definition 4.5: Solving a Linear System

The process of solving a linear system consists of finding all possible vectors $X = (x_1, x_2, \ldots, x_n)$ that satisfy the matrix equation $AX = B$, where:

- A is an $m \times n$ coefficient matrix,
- X is an $n \times 1$ column vector of variables,
- B is an $m \times 1$ column vector of constants.

4.2.1 Solving by Cramer's Rule

Definition 4.6: Cramer System

A linear system (S) is called a Cramer system if:

- Its coefficient matrix A is square ($n \times n$),
- A is invertible (i.e., $\det(A) \neq 0$).

Theorem 4.1: Cramer's Rule

Let (S): $AX = B$ be a Cramer system where:

- $A = [C_1|\cdots|C_n]$ is the $n \times n$ coefficient matrix with column vectors $C_1, \ldots, C_n$,
- B is the right-hand side column vector.

The unique solution $X = (x_1, \ldots, x_n)^T$ is given by:

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad 1 \leq i \leq n,$$

where A_i is the matrix formed by replacing the i-th column of A with B:

$$A_i = [C_1|\cdots|C_{i-1}|B|C_{i+1}|\cdots|C_n].$$

Example 4.4: Cramer's Rule Application

Consider the system:

$$(S):\begin{cases} 3x+3y-2z=5, \\ 2y+7z=0, \\ x+y-z=3. \end{cases}$$

with matrix representation:

$$A = \begin{pmatrix} 3 & 3 & -2 \\ 0 & 2 & 7 \\ 1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}.$$

1. Verify Cramer condition:

$$\det(A) = 3\begin{vmatrix} 2 & 7 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 0 & 7 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}$$
$$= 3(-9) - 3(-7) - 2(-2) = -27 + 21 + 4 = -2 \neq 0.$$

Thus (S) is a Cramer system.

2. Compute solutions:

$$x = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 0 & 2 & 7 \\ 3 & 1 & -1 \end{vmatrix}}{\det(A)} = \frac{-30}{-2} = 15,$$

$$y = \frac{\begin{vmatrix} 3 & 5 & -2 \\ 0 & 0 & 7 \\ 1 & 3 & -1 \end{vmatrix}}{\det(A)} = \frac{-28}{-2} = 14,$$

$$z = \frac{\begin{vmatrix} 3 & 3 & 5 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{vmatrix}}{\det(A)} = \frac{8}{-2} = -4.$$

The unique solution is $(x, y, z) = (15, 14, -4)$.

4.3 Solving by Gaussian Elimination

The following operations can be performed on any matrix to obtain an equivalent system :

Definition 4.7: Elementary Row Operations

1. **Row Replacement:** Replace row i with the sum of itself and a nonzero multiple of row j:
$$R_i \to R_i + cR_j \quad (i \neq j, c \neq 0).$$

2. **Row Interchange:** Swap two distinct rows:
$$R_i \leftrightarrow R_j \quad (i \neq j).$$

3. **Row Scaling:** Multiply all entries in a row by a nonzero constant:
$$R_i \to cR_i \quad (c \neq 0).$$

Remark 4.1: Properties of Elementary Row Operations

These operations:

- Preserve the solution set of the corresponding linear system,
- Are reversible (each operation has an inverse operation),
- Are used in Gaussian elimination to obtain row echelon form.

4.3.1 Gaussian Elimination Method

Definition 4.8: Gaussian Elimination

Gaussian elimination is a systematic procedure for solving systems of linear equations using elementary row operations to transform the augmented matrix into row echelon form (REF).

Algorithm 4.1: Gaussian Elimination Steps

1. **Forward Elimination:**
   (a) Start with the leftmost nonzero column (pivot column),
   (b) Select a nonzero entry (pivot) in this column, preferably 1,
   (c) Move the pivot row to the top position if needed (Row Interchange),
   (d) Create zeros below the pivot using row replacement operations,
   (e) Repeat for each subsequent pivot position, moving right and down.
2. **Back Substitution:**
   (a) Starting from the last nonzero row, solve for the leading variable,
   (b) Substitute this value into the rows above,
   (c) Repeat upward through all rows.

Example 4.5: Gaussian Elimination - Unique Solution

Solve the system:

$$\begin{cases} x + 2y + z = 4, \\ 2x + 5y - z = 5, \\ x + 3y + 4z = 7. \end{cases}$$

1. Augmented matrix:

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & 5 & -1 & 5 \\ 1 & 3 & 4 & 7 \end{pmatrix}.$$

2. Forward elimination:

   - $R_2 \to R_2 - 2R_1$,
   - $R_3 \to R_3 - R_1$,

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 1 & 3 & 3 \end{pmatrix},$$

   - $R_3 \to R_3 - R_2$,

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 6 & 6 \end{pmatrix}.$$

3. Back substitution:

$$6z = 6 \Rightarrow z = 1,$$
$$y - 3(1) = -3 \Rightarrow y = 0,$$
$$x + 2(0) + 1 = 4 \Rightarrow x = 3.$$

Solution: $(3, 0, 1)$.

Example 4.6: Gaussian Elimination - Infinite Solutions

Solve:

$$\begin{cases} x + y + z = 3, \\ 2x + 2y + 4z = 8, \\ x + y + 2z = 4. \end{cases}$$

1. Augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 4 & 8 \\ 1 & 1 & 2 & 4 \end{pmatrix}.$$

2. Forward elimination:

   - $R_2 \to R_2 - 2R_1$,
   - $R_3 \to R_3 - R_1$,

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

   - $R_3 \to R_3 - \frac{1}{2}R_2$,

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3. Solution:

$$\begin{cases} x + y + z = 3, \\ 2z = 2, \end{cases} \Rightarrow \begin{cases} z = 1, \\ x = 3 - y - 1 = 2 - y. \end{cases}$$

General solution: $(2 - t, t, 1)$ where $t \in \mathbb{R}$.