3.6 Exercises

Exercise 3.1: Matrix Operations

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 4 \\ 2 & 3 & 0 \end{pmatrix}.$$

1. Compute the following:

$$A+B, \quad 3B, \quad -1B, \quad A+2B, \quad 2A+B, \quad A-B, \quad A-1B, \quad B-A.$$

2. Write down the row vectors and column vectors of the matrices A, B.
3. Find $A^t, B^t, (A+B)^t$ and $A^t + B^t$.

Exercise 3.2: Symmetric and Skew-Symmetric Matrices

1. Show that for any square matrix, the matrix $A + A^t$ is symmetric.
2. Define a matrix A to be skew-symmetric if $A^t = -A$. Show that for any square matrix A, the matrix $A - A^t$ is skew-symmetric.
3. If a matrix is skew-symmetric, what can you say about its diagonal elements?

Exercise 3.3: Matrix Multiplication

In each of the following cases, find $(AB)C$ and $A(BC)$ (if defined).

Case 1:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Case 2:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$

Case 3:

$$A = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Exercise 3.4: Matrix-Vector Multiplication

Let

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 5 \end{pmatrix}.$$

Find AX for each of the following values of X:

- $X = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$,
- $X = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$,
- $X = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

Exercise 3.5: Matrix Representation of Linear Maps

Write the matrices of the following linear transformations relative to the canonical bases:

1. $f_1 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f_1(x, y, z) = (x + 2y + 3z, 2y - z, x + z).$$

2. $f_2 : \mathbb{R}_2[X] \to \mathbb{R}_3[X]$ defined by

$$f_2(P) = XP - P' + P(1),$$

where $\mathbb{R}_n[X]$ denotes the space of polynomials with real coefficients of degree $\leq n$, and P' is the derivative of P.

Exercise 3.6: Matrix Inversion

Using two different methods, compute the inverses of the following matrices:

$$A = \begin{pmatrix} 1 & -3 & 5 \\ 1 & -1 & 2 \\ -1 & 2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Exercise 3.7: Matrix Polynomial and Inversion

Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}.$$

1. Compute $A^3 - A$.
2. Deduce that A is invertible and determine A^{-1}.

Exercise 3.8: Linear Map and Matrix Rank

Let f be the linear map defined by

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, z) \mapsto f(x, y, z) = (x + y, y + z).$$

1. Determine the matrix associated with f in the canonical bases.

2. Compute the rank of the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

Exercise 3.9: Linear Transformation

Let $B = (e_1, e_2)$ be the canonical basis of $\mathbb{R}^2$. Consider the linear map f from $\mathbb{R}^2$ defined by the matrix A in the basis B:

$$A = \begin{pmatrix} 11 & 30 \\ -11 & 4 \end{pmatrix}.$$

1. Determine the vectors $f(e_1)$, $f(e_2)$, $f(2,5)$, and $f(1,3)$.
2. Give the expression of the function f.

Exercise 3.10: Matrix Polynomial and Inversion

Let

$$A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. Compute A^2, then find two real numbers α and β such that

$$A^2 = \alpha A + \beta I.$$

2. Deduce from the above that A is invertible, and find A^{-1}. Then compute A^{-1} again using the comatrix method.

Exercise 3.11: Matrix Rank

Compute the rank of the following matrices:

1. $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$,

2. $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

3. $A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 3 \\ 2 & 2 & -2 \end{pmatrix}$,

4. $A = \begin{pmatrix} 8 & 4 & -16 \\ 0 & 4 & -8 \\ 4 & 4 & -12 \end{pmatrix}$.