Chapter 3

Matrices

Matrices are fundamental mathematical objects that provide a structured way to organize and manipulate data, particularly in linear algebra. A matrix is a rectangular array of elements (numbers, functions, or other mathematical objects) arranged in rows and columns. Matrices are not only powerful tools for solving systems of linear equations but also play a crucial role in representing linear transformations between vector spaces.

In this chapter, we explore the algebraic structure of matrices, including operations such as addition, scalar multiplication, and matrix multiplication. We also study special types of matrices, such as square matrices, identity matrices, and invertible matrices, along with their properties. Additionally, we examine the determinant and rank of a matrix, which are key concepts in determining invertibility and solving linear systems.

3.1 Definitions

Definition 3.1: Matrix

A matrix of type (p, n) with coefficients in the field $\mathbb{K}$ is a table A of $p \times n$ elements of $\mathbb{K}$, arranged in p rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{pmatrix}$$

or, abbreviated: $A = (a_{ik})$; or also: $A = \|a_{ik}\|$. The set of matrices with p rows and n columns is denoted by $M_{p,n}(\mathbb{K})$. If $n = p$, $M_{n,n}(\mathbb{K})$ is denoted $M_n(\mathbb{K})$.

Note that, in the notation we have adopted, a_{ik} denotes the element in the i-th row and the k-th column.

Example 3.1

Thus, for example:

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \end{pmatrix} \in M_{2,3}(\mathbb{R}), \quad \begin{pmatrix} 1 & 2-i & 3+i \\ 0 & 1+i & i \\ -i & 2 & 1 \end{pmatrix} \in M_3(\mathbb{C}).$$

On the set $M_{p,n}(\mathbb{K})$, we define the following operations:

Definition 3.2: Sum of Two Matrices

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices in $M_{p,n}(\mathbb{K})$. The sum $A + B$ is the matrix $(a_{ij} + b_{ij})$ in $M_{p,n}(\mathbb{K})$.

Example 3.2: Matrix Addition

1. If $A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 3 & 4 \\ 1 & 1 & -1 \end{pmatrix}$, then:

$$A + B = \begin{pmatrix} 1+(-2) & 1+3 & 0+4 \\ -2+1 & 3+1 & -1+(-1) \end{pmatrix} = \begin{pmatrix} -1 & 4 & 4 \\ -1 & 4 & -2 \end{pmatrix}.$$

2. If $A = \begin{pmatrix} -3 & 1 \\ 7 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 3 & 4 \\ 5 & -1 \end{pmatrix}$, then $A + B$ is not defined.

Definition 3.3: Scalar Multiplication of a Matrix

Let $A = (a_{ij}) \in M_{n,m}(\mathbb{K})$ and $\lambda \in \mathbb{K}$. The product λA is the matrix (λa_{ij}).

Example 3.3: Scalar Multiplication

If $A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 3 & -1 \end{pmatrix}$, then:

$$2A = \begin{pmatrix} 2\cdot 1 & 2\cdot 1 & 2\cdot 0 \\ 2\cdot(-2) & 2\cdot 3 & 2\cdot(-1) \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ -4 & 6 & -2 \end{pmatrix}.$$

Remark 3.1: Properties of Matrix Operations

1. The matrix $-A = (-1)A$ is the additive inverse of A.
2. $A - B = A + ((-1)B)$.

Theorem 3.1: Properties of Matrix Operations

Let A, B, and C be matrices of the same size, and let r and s be scalars. The following properties hold:

1. $A + B = B + A$. (Commutativity of addition)
2. $(A + B) + C = A + (B + C)$. (Associativity of addition)
3. $A + 0 = A$. (Existence of additive identity)
4. $r(A + B) = rA + rB$. (Distributivity of scalar over matrix sum)
5. $(r + s)A = rA + sA$. (Distributivity of scalar sum over matrix)
6. $r(sA) = (rs)A$. (Associativity of scalar multiplication)

It is easy to see that, equipped with these operations, $M_{p,n}(\mathbb{K})$ is a vector space over $\mathbb{K}$. The neutral element is the matrix where all elements are zero, called the zero matrix, and denoted by 0. The opposite of the matrix (a_{ik}) is the matrix $(-a_{ik})$.

Theorem 3.2: Vector Space of Matrices

The set $M_{p,n}(\mathbb{K})$, equipped with addition and scalar multiplication, forms a $\mathbb{K}$-vector space.

Proposition 3.1: Dimension of Matrix Space

$$\dim_{\mathbb{K}} M_{p,n}(\mathbb{K}) = pn.$$

Indeed, it is straightforward to verify that the pn matrices, called elementary matrices:

$$E_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \ldots, \quad E_{ik} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

$$(1 \text{ in position } (i,k)), \quad \ldots, \quad E_{pn} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix},$$

form a basis of $M_{p,n}(\mathbb{K})$, called the canonical basis.

3.2 Matrices Associated with Linear Maps

Let E and E' be two vector spaces over $\mathbb{K}$ of dimensions n and p respectively, and $f : E \to E'$ a linear map. Choose a basis $\{e_1, \ldots, e_n\}$ for E and a basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$ for E'. The images under f of the vectors $e_1, \ldots, e_n$ can be expressed in terms of the basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$:

$$\begin{aligned} f(e_1) &= a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{p1}\varepsilon_p \\ f(e_2) &= a_{12}\varepsilon_1 + a_{22}\varepsilon_2 + \cdots + a_{p2}\varepsilon_p \\ &\vdots \\ f(e_n) &= a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{pn}\varepsilon_p \end{aligned}$$

Definition 3.4: Matrix of a Linear Map

The matrix of f with respect to the bases $\{e_1, \ldots, e_n\}$ and $\{\varepsilon_1, \ldots, \varepsilon_p\}$ is the matrix denoted $M(f)_{e_i,\varepsilon_j}$ in $M_{p,n}(\mathbb{K})$ whose columns are the components of the vectors

$f(e_1), \dots, f(e_n)$ in the basis $\{\varepsilon_1, \dots, \varepsilon_p\}$:

$$M(f)_{e_i,\varepsilon_j} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{pmatrix}$$

If there is no ambiguity, we may simply write $M(f)$ instead of $M(f)_{e_i,\varepsilon_j}$, but it is clear that the matrix associated with f depends on the choice of bases for E and E'. In the case where f is an endomorphism, we can choose the same basis for E considered as both the departure and arrival space. In this case, we will write $M(f)_{e_i}$ instead of $M(f)_{e_i,e_j}$.

Proposition 3.2: Matrix Representation Isomorphism

Let E and E' be two vector spaces over $\mathbb{K}$ of dimensions n and p respectively, with bases $\{e_i\}$ and $\{\varepsilon_j\}$ for E and E'. Then the mapping:

$$\begin{aligned} M : \mathcal{L}_{\mathbb{K}}(E, E') &\to M_{p,n}(\mathbb{K}) \\ f &\mapsto M(f)_{e_i,\varepsilon_j} \end{aligned}$$

is a vector space isomorphism, that is:

$$\begin{cases} M(f+g) = M(f) + M(g) \\ M(\lambda f) = \lambda M(f) \end{cases}$$

and M is bijective.
In particular: $\dim_{\mathbb{K}} \mathcal{L}(E, E') = np$.

Proof 3.1: I

deed, we have:

$$\begin{aligned} M(f+g)_{e_i,\varepsilon_j} &= \|(f+g)(e_1), \dots, (f+g)(e_n)\|_{\varepsilon_j} \\ &= \|f(e_1) + g(e_1), \dots, f(e_n) + g(e_n)\|_{\varepsilon_j} \\ &= \|f(e_1), \dots, f(e_n)\|_{\varepsilon_j} + \|g(e_1), \dots, g(e_n)\|_{\varepsilon_j} \end{aligned}$$

by the definition of matrix addition, that is:

$$M(f+g)_{e_i,\varepsilon_j} = M(f)_{e_i,\varepsilon_j} + M(g)_{e_i,\varepsilon_j}.$$

Similarly, if $\lambda \in \mathbb{K}$:

$$\begin{aligned} M(\lambda f)_{e_i,\varepsilon_j} &= \|(\lambda f)(e_1), \dots, (\lambda f)(e_n)\|_{\varepsilon_j} \\ &= \|\lambda f(e_1), \dots, \lambda f(e_n)\|_{\varepsilon_j} \\ &= \lambda \|f(e_1), \dots, f(e_n)\|_{\varepsilon_j} \end{aligned}$$

Thus M is linear.
Moreover, M is surjective. Indeed, let:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pn} \end{pmatrix} \in M_{p,n}(\mathbb{K})$$

and define $f \in \mathcal{L}(E, E')$ as follows. First set:

$$\begin{aligned} f(e_1) &= a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{p1}\varepsilon_p \\ &\vdots \\ f(e_n) &= a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{pn}\varepsilon_p \end{aligned}$$

Then extend f linearly to E, i.e., for:

$$x = \lambda_1 e_1 + \cdots + \lambda_n e_n \in E, \quad \text{define:} \quad f(x) = \lambda_1 f(e_1) + \cdots + \lambda_n f(e_n).$$

It is easy to verify that f is linear and that $A = M(f)_{e_i,\varepsilon_j}$.
Finally, M is injective. Indeed, let $f \in \ker M$:

$$M(f) = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

This means that $f(e_1) = 0, \dots, f(e_n) = 0$. Therefore, for any $x = \lambda_1 e_1 + \cdots + \lambda_n e_n \in E$, we will have:

$$f(x) = \lambda_1 f(e_1) + \cdots + \lambda_n f(e_n) = 0,$$

that is, $f = 0$. By Proposition 3.5, f is injective.

Example 3.4: Matrix Representations

1. Let E be a space of dimension n and consider the identity map:

$$\begin{aligned} \mathrm{id}_E : E &\to E \\ x &\mapsto x \end{aligned}$$

Given a basis $\{e_i\}$, we have $\mathrm{id}_E(e_k) = e_k$. Therefore:

$$M(\mathrm{id}_E)_{e_i} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

(1 is the unit element of $\mathbb{K}$). This matrix is denoted I_n or simply I and is called the identity matrix of $M_n(\mathbb{K})$.

2. Let $E = \mathbb{R}^2$ and consider the projection:

$$\begin{aligned} \mathrm{pr}_1 : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (x, 0) \end{aligned}$$

Using the canonical basis of $\mathbb{R}^2$, we have:

$$\begin{cases} \mathrm{pr}_1(e_1) = e_1 \\ \mathrm{pr}_1(e_2) = 0 \end{cases}$$

Thus:

$$M(\mathrm{pr}_1)_{e_i} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

3. Let $E = \mathbb{R}^2$ and f be the projection onto the first bisector parallel to the second bisector. With $\{e_1, e_2\}$ being the canonical basis of E, we have:

$$f(e_1) = f(e_2) = \varepsilon = \frac{1}{2}(e_1 + e_2)$$

Therefore:

$$M(f)_{e_i} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

4. Let $E = \mathbb{R}^2$ and f be the reflection about the Ox axis parallel to the Oy axis. With the canonical basis $\{e_1, e_2\}$, we have:

$$\begin{cases} f(e_1) = e_1 \\ f(e_2) = -e_2 \end{cases}$$

Thus:

$$M(f)_{e_i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3.3 Matrix Multiplication

Definition 3.5: Matrix Multiplication

Let $A \in M_{m,n}(\mathbb{K})$ be an $m \times n$ matrix and $B \in M_{n,p}(\mathbb{K})$ an $n \times p$ matrix. The matrix product AB is defined as the $m \times p$ matrix whose (i, j)-entry is given by:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

This operation is called matrix multiplication.

Remark 3.2: Matrix Multiplication Condition

Matrix multiplication is only defined when the number of columns of A matches the number of rows of B.

Example 3.5: Matrix Multiplication Calculation

Consider the matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in M_{2,3}(\mathbb{R}), \quad B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} \in M_{3,2}(\mathbb{R}).$$

The product AB is computed as follows:

$$\begin{aligned} AB &= \begin{pmatrix} (1\cdot 7 + 2\cdot 9 + 3\cdot 11) & (1\cdot 8 + 2\cdot 10 + 3\cdot 12) \\ (4\cdot 7 + 5\cdot 9 + 6\cdot 11) & (4\cdot 8 + 5\cdot 10 + 6\cdot 12) \end{pmatrix} \\ &= \begin{pmatrix} 7 + 18 + 33 & 8 + 20 + 36 \\ 28 + 45 + 66 & 32 + 50 + 72 \end{pmatrix} \\ &= \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}. \end{aligned}$$

Thus, $AB \in M_{2,2}(\mathbb{R})$.

Theorem 3.3: Properties of Matrix Multiplication

For compatible matrices A, B, C and scalar λ:

1. Associativity: $(AB)C = A(BC)$.
2. Distributivity: $A(B + C) = AB + AC$.
3. Scalar compatibility: $\lambda(AB) = (\lambda A)B = A(\lambda B)$.
4. Identity: $I_m A = A = AI_n$ where I is the identity matrix.
5. $A \cdot 0 = 0$ and $0 \cdot A = 0$.

Example 3.6: Matrix Multiplication Examples

1. Let

$$A = \begin{pmatrix} -1 & 3 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 4 & -1 \\ 0 & -1 & -2 & 1 \end{pmatrix}.$$

Since $A \in M_{3,2}(\mathbb{R})$ and $B \in M_{2,4}(\mathbb{R})$, the product AB is defined and

$AB \in M_{3,4}(\mathbb{R})$. Moreover:

$$AB = \begin{pmatrix} -1 & -4 & -10 & 4 \\ 0 & -1 & -2 & 1 \\ 2 & 1 & 6 & -1 \end{pmatrix}.$$

2. Consider the matrices

$$A = \begin{pmatrix} 1 & i \\ 2 & -i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -i \\ 0 & 2i \end{pmatrix}.$$

Since $A \in M_2(\mathbb{C})$ and $B \in M_2(\mathbb{C})$, the product AB is defined and $AB \in M_2(\mathbb{C})$. Moreover:

$$\begin{aligned} AB &= \begin{pmatrix} 3 - i + i(2i) & -i + i(2i) \\ 6 - 2i - i(2i) & -2i - i(2i) \end{pmatrix} \\ &= \begin{pmatrix} 3 - i - 2 & -i - 2 \\ 6 - 2i + 2 & -2i + 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 - i & -2 - i \\ 8 - 2i & 2 - 2i \end{pmatrix}. \end{aligned}$$

3. If

$$A = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 5 \\ 1 & 3 \\ 1 & -1 \end{pmatrix},$$

then AB is not defined because $A \in M_{2,1}(\mathbb{R})$ and $B \in M_{3,2}(\mathbb{R})$ (the number of columns of A doesn't match the number of rows of B).

Remark 3.3: Important Remarks on Matrix Multiplication

1. If $A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$, then

$$AB = \begin{pmatrix} 4 & -1 & -3 \\ -6 & 2 & 4 \end{pmatrix},$$

but BA is not defined. Therefore, in general, $AB \neq BA$.

2. Since
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
we conclude that $AB = 0$ does not imply $A = 0$ or $B = 0$.

3. From the fact that
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$
we see that $AB = AC$ does not imply $B = C$.

Remark 3.4: Computing the Image of a Vector via a Matrix Representation

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and A its associated matrix (relative to the standard bases). If v is a vector in $\mathbb{R}^n$, its image $f(v)$ is computed by:

$$f(v) = Av,$$

where the operation is standard matrix-vector multiplication.

Example 3.7: Matrix-Vector Multiplication

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ be the matrix of the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$. To compute the image of the vector $v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$:

$$f(v) = Av = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} (1)(2) + (2)(-1) \\ (3)(2) + (4)(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Thus, $f(v) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

3.4 Matrix Transpose

Definition 3.6: Matrix Transpose

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The transpose of A, denoted A^T, is the $n \times m$ matrix obtained by interchanging the rows and columns of A:

$$(A^T)_{ij} = a_{ji} \quad \text{for all } i, j.$$

That is, the ith row of A becomes the ith column of A^T.

Example 3.8: Matrix Transpose

For the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

its transpose is:

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

Remark 3.5: Key Properties of Transpose

- $(A^T)^T = A$ (Involution).
- $(A + B)^T = A^T + B^T$ (Linearity).
- $(AB)^T = B^T A^T$ (Reverse-order law).
- $(\lambda A)^T = \lambda A^T$ for scalar λ.

Definition 3.7: Special Matrix Types

- A matrix is *symmetric* if $A^T = A$.
- A matrix is *skew-symmetric* if $A^T = -A$.
- A matrix is *orthogonal* if $A^T = A^{-1}$.

Example 3.9: Special Matrix Types

1. **Symmetric Matrix** ($A^T = A$):

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}; \quad A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}.$$

   Note: All diagonal matrices are symmetric.

2. **Skew-Symmetric Matrix** ($A^T = -A$):

$$B = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 4 \\ 1 & -4 & 0 \end{pmatrix}; \quad B^T = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -4 \\ -1 & 4 & 0 \end{pmatrix} = -B.$$

   Note: Diagonal entries must be zero.

3. **Orthogonal Matrix** ($A^T = A^{-1}$):

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \quad Q^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = Q^{-1}.$$

   Verification: $QQ^T = I_2$ (rotation matrices are orthogonal).

3.5 Square Matrices

Definition 3.8: Square Matrix

A square matrix is a matrix with the same number of rows and columns, i.e., a matrix of size $n \times n$ for some positive integer n. The set of all $n \times n$ matrices over a field $\mathbb{K}$ is denoted by $M_n(\mathbb{K})$.

Definition 3.9: Identity Matrix

The square matrix

$$I_n := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called the identity matrix.

Remark 3.6: Properties of Identity Matrix

Let $A \in M_{n,m}(\mathbb{K})$. Then:

$$I_n A = A \quad \text{and} \quad A I_m = A.$$

Example 3.10: Identity Matrix Multiplication

Let $A \in M_{2,3}(\mathbb{R})$ be the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

and let I_2 be the identity matrix of size 2×2:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then:

$$I_2 A = A \quad \text{and} \quad A I_3 = A.$$

3.5.1 Determinant of a Square Matrix

Definition 3.10: Submatrices

Let $A = (a_{ij}) \in M_n(\mathbb{K})$. We denote by $A_{\setminus i,j}$ the submatrix of A of order $n-1$ obtained by removing the i-th row and j-th column from A.

Example 3.11: Submatrices Examples

1. For the matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 5 & -2 & 0 \end{pmatrix}$, we have:

$$A_{\setminus 1,1} = \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix}; \quad A_{\setminus 2,2} = \begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix}; \quad A_{\setminus 3,2} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$$

2. For the matrix $B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 4 \\ 0 & 5 & 1 \end{pmatrix}$, we get:

$$B_{\setminus 1,3} = \begin{pmatrix} 3 & -1 \\ 0 & 5 \end{pmatrix}; \quad B_{\setminus 2,1} = \begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix}; \quad B_{\setminus 3,3} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}.$$

Definition 3.11: Determinant

Let $A = (a_{ij}) \in M_n(\mathbb{K})$. The determinant of A, denoted by $\det(A)$ or $|A|$, is the element of $\mathbb{K}$ defined recursively by:

(i) If $n = 1$, then $\det(A) = a_{11}$.

(ii) If $n > 1$, then

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{\setminus 1j}),$$

where $A_{\setminus 1j}$ is the submatrix obtained by deleting the first row and j-th column of A.

Example 3.12: Determinant Calculation

- 2×2 Case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

- 3×3 Case (Sarrus' Rule): Sarrus' rule provides a simple mnemonic method for computing the determinant of a 3×3 matrix without using cofactor expansion.

The Rule

Given a 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

1. Write the first two columns again to the right of the matrix:

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

2. Then the determinant is:

$$\begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32}). \end{aligned}$$

Example 3.13: Sarrus' Rule Application

Compute the determinant of:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

1. Extend the matrix:

$$\begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \\ 7 & 8 & 9 & 7 & 8 \end{array}$$

2. Calculate the products:

$$1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 = 45 + 84 + 96 = 225$$

$$3 \cdot 5 \cdot 7 + 2 \cdot 4 \cdot 9 + 1 \cdot 6 \cdot 8 = 105 + 72 + 48 = 225$$

3. Final determinant:

$$\det(B) = 225 - 225 = 0$$

Remark 3.7: Determinant Expansion Methods

The determinant can be computed using expansion along any row or column:

1. **Row Expansion:** For any row i:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{\setminus ij}),$$

where $A_{\setminus ij}$ is the submatrix obtained by deleting row i and column j.

2. **Column Expansion:** For any column j:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{\setminus ij}).$$

3. **Optimal Strategy:** For efficient computation, choose the row/column with the most zeros.

Example 3.14: Determinant Expansion with Zeros

For matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, expanding along the second column (which has two zeros) gives:

$$\begin{aligned} \det(A) &= (-1)^{1+2} \cdot 0 \cdot \det \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \\ &+ (-1)^{2+2} \cdot 0 \cdot \det \begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix} \\ &+ (-1)^{3+2} \cdot 6 \cdot \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ &= -6 \cdot (1 \cdot 4 - 2 \cdot 3) = -6 \cdot (4 - 6) = -6 \cdot (-2) = 12. \end{aligned}$$

Example 3.15: Determinant Calculations

1. $$\det\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix} = (2\cdot 4) - (3\cdot 1) = 8 - 3 = 5.$$

2. For a 3×3 matrix $A = \begin{pmatrix}0 & 4 & 5\\ 1 & 0 & 6\\ 2 & 1 & 7\end{pmatrix}$:

$$\begin{aligned}\det(A) &= 0\cdot\det\begin{pmatrix}0 & 6\\ 1 & 7\end{pmatrix} - 4\cdot\det\begin{pmatrix}1 & 6\\ 2 & 7\end{pmatrix} + 5\cdot\det\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\\ &= 0 - 4\cdot(1\cdot 7 - 6\cdot 2) + 5\cdot(1\cdot 1 - 0\cdot 2)\\ &= -4\cdot(7-12) + 5\cdot(1-0)\\ &= -4\cdot(-5) + 5\cdot 1 = 20 + 5 = 25.\end{aligned}$$

Theorem 3.4: Properties of Determinants

Let $A, B \in M_n(\mathbb{K})$. Then the following properties hold:

1. **Transpose Invariance:**
$$\det(A) = \det(A^T).$$

2. **Product Rule:**
$$\det(AB) = \det(A)\det(B).$$

3. **Scalar Multiplication:**
$$\det(\lambda A) = \lambda^n \det(A).$$

4. **Invertibility Criterion:** A is invertible if and only if $\det(A) \neq 0$.

5. **Inverse Matrix:**
$$\text{If } A \text{ is invertible, then } \det(A^{-1}) = \frac{1}{\det(A)}.$$

Example 3.16: Determinant Properties Examples

1. **Product Rule:**

$$\text{If } A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$\det(AB) = \det \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} = -6 = 6 \cdot (-1).$$

2. **Scalar Multiplication:**

$$\text{For } A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \lambda = 2; \quad \det(2A) = 4 \cdot 0 = 0.$$

3. **Invertibility Criterion:**

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ is singular (det = 0);} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ is invertible (det = } -2).$$

4. **Inverse Matrix:**

$$\text{For } A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}; \quad \det(A) = 8; \quad \det(A^{-1}) = \frac{1}{8}.$$

Example 3.17: Fundamental Determinant Properties Examples

1. **Zero Row Property:** Consider the matrix:

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{pmatrix}.$$

The second row is entirely zero, thus $\det(A) = 0$.

2. **Proportional Rows:** For the matrix:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Row 2 is exactly twice Row 1, hence $\det(B) = 0$.

3. **Row Interchange:** Compare:

$$C = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}; \quad C' = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}.$$

We find:

$$\det(C) = -10; \quad \det(C') = 10 = -\det(C).$$

4. **Scalar Multiplication:** Original matrix and scaled version:

$$D = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}; \quad D' = \begin{pmatrix} 6 & 3 \\ 1 & 3 \end{pmatrix}.$$

Determinants verify:

$$\det(D) = 5; \quad \det(D') = 15 = 3 \cdot \det(D).$$

5. **Row Addition:** Original and modified matrices:

$$E = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}; \quad E' = \begin{pmatrix} 1+3 & 5+2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 3 & 2 \end{pmatrix}.$$

Both have:

$$\det(E) = \det(E') = -13.$$

3.5.2 Invertible matrix

Matrix algebra provides tools for manipulating matrix equations and creating various useful formulas in ways similar to doing ordinary algebra with real numbers. This subsection investigates the matrix analogue of the reciprocal, or multiplicative inverse, of a nonzero number.

Definition 3.12: Invertible Matrix

Let $A \in M_n(\mathbb{K})$. We say that A is invertible if there exists a matrix $B \in M_n(\mathbb{K})$ such that:

$$AB = BA = I_n.$$

The matrix B is called the inverse matrix of A and is denoted by A^{-1}.

Proposition 3.3: Uniqueness of Inverse

If an inverse exists, then there is only one (we say that the inverse is unique, or uniquely determined by A).

Proof 3.2: S

ppose that B and C are both inverses of A, so we have:

$$AB = BA = I \quad \text{and} \quad AC = CA = I.$$

We want to show that $B = C$.
Multiply the equation $BA = I$ on the right by C:

$$BAC = IC = C.$$

Since $AC = I$, we have:

$$BAC = BI = B.$$

Therefore, $B = C$.
This proves that the inverse is unique. Thus, the inverse of A, denoted A^{-1}, is the unique matrix such that:

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

Example 3.18: Invertible Matrices

1. The matrix I_n is invertible and its inverse is I_n, because $I_n^2 = I_n$.

2. The matrix

$$A = \begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix},$$

is invertible and its inverse is the matrix

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 3 & -5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{11} & -\frac{5}{11} \\ \frac{1}{11} & \frac{2}{11} \end{pmatrix}.$$

Theorem 3.5: Properties of Matrix Inverses

1. If A is an invertible matrix, then A^{-1} is invertible and:

$$(A^{-1})^{-1} = A.$$

2. If A and B are $n \times n$ invertible matrices, then AB is invertible, and the inverse of AB is the product of the inverses of A and B in reverse order:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

3. If A is an invertible matrix, then its transpose A^T is also invertible, and the inverse of A^T is the transpose of A^{-1}:

$$(A^T)^{-1} = (A^{-1})^T.$$

Proof 3.3: Properties of Matrix Inverses

1. To verify statement (a), we must find a matrix C such that:

$$A^{-1}C = I \quad \text{and} \quad CA^{-1} = I.$$

These equations are satisfied when we take $C = A$. Hence, A^{-1} is invertible, and its inverse is A.

2. To prove statement (b), compute:

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I, \\ (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I. \end{aligned}$$

Therefore, AB is invertible, and:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

3. To verify statement (c), we use Theorem 3(d), read from right to left:

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I,$$
$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Therefore, A^T is invertible, and its inverse is:

$$(A^T)^{-1} = (A^{-1})^T.$$

Definition 3.13: General Linear Group

The general linear group of degree n over a field $\mathbb{K}$, denoted by $\mathrm{GL}_n(\mathbb{K})$, is the group of all invertible elements of $M_n(\mathbb{K})$.

Theorem 3.6: Ring Structure of $M_n(\mathbb{K})$

The set $M_n(\mathbb{K})$, equipped with matrix addition and multiplication, forms a ring. The identity element for multiplication is the identity matrix I_n.
For $n > 1$, the ring $M_n(\mathbb{K})$ is non-commutative and not an integral domain.

Definition 3.14: Elementary Operations on a Matrix

Elementary operations on a matrix A are defined as:

1. Swapping two rows (respectively, two columns) of A.
2. Adding to a row (respectively, a column) r another row (respectively, column) s, with $s \neq r$, multiplied by a scalar $\lambda \in \mathbb{K}$.
3. Multiplying a row (respectively, a column) of A by a nonzero scalar $\lambda \in \mathbb{K} \setminus \{0\}$.

Algorithm 3.1: Matrix Inversion Algorithm

To compute the inverse of a matrix A of order n, knowing that A is invertible, we proceed as follows:

1. Consider the matrix E of size $n \times 2n$, where the first n columns are those of A, and the last n columns are those of the identity matrix I_n.

2. Apply a sequence of elementary row operations to E so that the first n columns are transformed into the identity matrix I_n. At this point, the last n columns of the resulting matrix form the inverse of A.

Example 3.19: Finding the Inverse of a Matrix Using Row Operations

Let

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}.$$

We want to compute A^{-1} using elementary row operations.

Step 1: Write the augmented matrix $[A|I]$:

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{pmatrix}.$$

Step 2: Make the pivot in row 1 equal to 1: Divide row 1 by 2:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 5 & 3 & 0 & 1 \end{pmatrix}.$$

Step 3: Eliminate below the pivot: Row 2 = Row 2 - 5 × Row 1:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{pmatrix}.$$

Step 4: Make the pivot in row 2 equal to 1: Multiply row 2 by 2:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -5 & 2 \end{pmatrix}.$$

Step 5: Eliminate above the pivot in row 1: Row 1 = Row 1 - $\frac{1}{2}$ × Row 2:

$$\begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{pmatrix}.$$

Result: The inverse of A is:

$$A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.$$

Definition 3.15: Cofactor Matrix

Let $A = (a_{ij})$ be a square matrix of order n. The *cofactor* associated with the entry a_{ij} is the number

$$(-1)^{i+j} \det(A^*_{i,j}),$$

where $A^*_{i,j}$ denotes the submatrix obtained by deleting the i-th row and the j-th column.

The *cofactor matrix* (or *adjugate matrix*) of A, denoted com(A) or adj(A), is the square matrix of order n whose entries are the corresponding cofactors.

Example 3.20: Cofactor Matrix Calculation

Consider the 3×3 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}.$$

The cofactor matrix com(A) is calculated as follows:

1. For $a_{11} = 1$:

$$C_{11} = (-1)^{1+1} \det \begin{pmatrix} 4 & 5 \\ 0 & 6 \end{pmatrix} = 1 \cdot (24 - 0) = 24.$$

2. For $a_{12} = 2$:

$$C_{12} = (-1)^{1+2} \det \begin{pmatrix} 0 & 5 \\ 1 & 6 \end{pmatrix} = -1 \cdot (0 - 5) = 5.$$

3. For $a_{13} = 3$:

$$C_{13} = (-1)^{1+3} \det \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} = 1 \cdot (0 - 4) = -4.$$

4. Continuing this process for all elements: The complete cofactor matrix is:

$$\mathrm{com}(A) = \begin{pmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{pmatrix}.$$

Using the above notations, we can state the result concerning the computation of the inverse matrix.

Theorem 3.7: Matrix Inversion

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

where $\operatorname{adj}(A)$ denotes the adjugate (the transpose of the cofactor matrix) of A.

Example 3.21: Matrix Inverse via Adjugate Method

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Step 1: Compute cofactors:

$$\begin{aligned} C_{11} &= (-1)^{1+1} \det(4) = 4, \\ C_{12} &= (-1)^{1+2} \det(3) = -3, \\ C_{21} &= (-1)^{2+1} \det(2) = -2, \\ C_{22} &= (-1)^{2+2} \det(1) = 1. \end{aligned}$$

Step 2: Build cofactor matrix:

$$\mathrm{com}(A) = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}.$$

Step 3: Transpose to get adjugate:

$$\operatorname{adj}(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

Step 4: Compute determinant:

$$\det(A) = (1)(4) - (2)(3) = -2.$$

Step 5: Apply inverse formula:

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

3.5.3 Matrix rank

Definition 3.16: Matrix Rank

Let $A \in M_{n,m}(\mathbb{K})$ be a matrix over a field $\mathbb{K}$. The rank of matrix A, denoted rank(A), is the dimension of the vector subspace of $\mathbb{K}^n$ generated by its column vectors:

$$\text{rank}(A) = \dim \text{Span}\{C_1, \dots, C_m\},$$

where $C_1, \dots, C_m$ are the columns of A.

Proposition 3.4: Properties of Matrix Rank

For any matrix $A \in M_{n,m}(\mathbb{K})$:

1. $\text{rank}(A) \leq \min(n, m)$.
2. $\text{rank}(A) = \text{rank}(A^T)$ (column rank equals row rank).
3. If B is invertible, $\text{rank}(AB) = \text{rank}(A)$.

Algorithm 3.2: Matrix Rank Calculation (Gaussian Elimination)

The rank of a matrix can be determined by performing row reduction to echelon form. The number of non-zero pivots in the echelon form is equal to the rank of the matrix.

Example 3.22: Matrix Rank Calculation

Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Step 1: Gaussian Elimination

We transform A to Row Echelon Form (REF) using elementary row operations.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 4R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{pmatrix}$$
$$\xrightarrow{R_3 \to R_3 - 7R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}.$$

Step 2: Determining the Rank

The REF shows:

- 2 non-zero rows $\Rightarrow$ rank is 2.
- The third row is all zeros, indicating linear dependence.

Step 3: Conclusion

The rank of matrix A is: $\text{rank}(A) = 2$.

Theorem 3.8: 3.42

Let $A \in M_n(\mathbb{K})$ be a square matrix of order n over a field $\mathbb{K}$. The following statements are equivalent:

1. A is invertible.
2. $\det(A) \neq 0$.
3. The rank of A equals n (full rank matrix).
4. The column vectors of A are linearly independent.

5. The row vectors of A are linearly independent.

6. The canonically associated linear map is bijective.