

2.5 Exercise Solutions

Solution 2.1

1. Let $f(x, y) = (2x + 3y, x)$. We check linearity:

$$\begin{aligned} f(x_1 + x_2, y_1 + y_2) &= (2(x_1 + x_2) + 3(y_1 + y_2), x_1 + x_2) \\ &= (2x_1 + 2x_2 + 3y_1 + 3y_2, x_1 + x_2) = f(x_1, y_1) + f(x_2, y_2). \end{aligned}$$

$$f(\lambda x, \lambda y) = (2\lambda x + 3\lambda y, \lambda x) = \lambda f(x, y).$$

Thus, f is linear.

To find the kernel:

$$f(x, y) = (0, 0) \Rightarrow \begin{cases} 2x + 3y = 0, \\ x = 0 \end{cases} \Rightarrow x = 0 \Rightarrow y = 0.$$

So, $\ker f = \{(0, 0)\}$, and a basis is \emptyset (or $\{0\}$).

To find the image, we compute the image of a basis of \mathbb{R}^2 .

Let $B_E = \{e_1 = (1, 0), e_2 = (0, 1)\}$.

Then:

$$f(e_1) = f(1, 0) = (2, 1), \quad f(e_2) = f(0, 1) = (3, 0).$$

Let us check whether these two vectors are linearly independent. Assume $\alpha(2, 1) + \beta(3, 0) = (0, 0)$:

$$(2\alpha + 3\beta, \alpha) = (0, 0) \Rightarrow \begin{cases} 2\alpha + 3\beta = 0, \\ \alpha = 0 \end{cases} \Rightarrow \beta = 0.$$

So, the vectors $(2, 1)$ and $(3, 0)$ are linearly independent. Therefore, they form a basis of $\text{Im}(f)$.

Since there are two linearly independent vectors in \mathbb{R}^2 , we have:

$$\dim(\text{Im}(f)) = 2 \Rightarrow \text{Im}(f) = \mathbb{R}^2.$$

So, a basis of the image is $\{(2, 1), (3, 0)\}$.

2. $f(x, y) = (y, x + y + 1)$. Check linearity:

$$f(0, 0) = (0, 1) \neq (0, 0).$$

So, f is not linear.

3. $f(x, y, z) = x + 2y + z$. This is a linear form, since it is a linear combination of the inputs.

To find the kernel of f , we solve:

$$f(x, y, z) = 0 \Rightarrow x + 2y + z = 0.$$

Let $y = s$ and $z = t$, where $s, t \in \mathbb{R}$ are free parameters. Then:

$$x = -2s - t.$$

So, the general solution of the kernel is:

$$(x, y, z) = (-2s - t, s, t) = s(-2, 1, 0) + t(-1, 0, 1).$$

Hence, the kernel is the span of two linearly independent vectors:

$$\ker f = \text{Span}\{(-2, 1, 0), (-1, 0, 1)\}.$$

Therefore, a basis of the kernel is:

$$\{(-2, 1, 0), (-1, 0, 1)\}.$$

The image is \mathbb{R} , since any real number can be written as a linear combination of $x + 2y + z$. A basis of the image is $\{1\}$.

4. $f(x, y) = (x + y, xy)$. Check linearity:

$$f(1, 1) = (2, 1), \quad f(2, 0) = (2, 0), \quad f(1, 1) + f(2, 0) = (4, 1),$$

while

$$f((1, 1) + (2, 0)) = f(3, 1) = (4, 3) \neq (4, 1).$$

So, f is not linear.

5. $f(x) = x^2$. This is not linear, since:

$$f(x + y) = (x + y)^2 = x^2 + 2xy + y^2 \neq f(x) + f(y).$$

So, f is not linear.

Solution 2.2

Consider the linear map:

$$f : \mathbb{R}[X] \rightarrow \mathbb{R}[X], \quad P \mapsto P + XP',$$

where P' denotes the derivative of P .

1. Linearity of f :

Let $P(X), Q(X) \in \mathbb{R}[X]$ and let $\lambda \in \mathbb{R}$. We have:

$$\begin{aligned} f(P + Q) &= (P + Q) + X(P + Q)' \\ &= P + Q + X(P' + Q') \\ &= (P + XP') + (Q + XQ') \\ &= f(P) + f(Q), \end{aligned}$$

and

$$f(\lambda P) = \lambda P + X(\lambda P)' = \lambda P + \lambda XP' = \lambda(P + XP') = \lambda f(P).$$

Hence, f is linear.

2. Kernel of f :

We want to find all polynomials P such that:

$$f(P) = P + XP' = 0.$$

Write $P(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$. Then:

$$P'(X) = a_1 + 2a_2X + \cdots + na_nX^{n-1},$$

and

$$XP'(X) = a_1X + 2a_2X^2 + \cdots + na_nX^n.$$

So:

$$\begin{aligned} P + XP' &= a_0 + (a_1 + a_1)X + (a_2 + 2a_2)X^2 + \cdots + (a_n + na_n)X^n \\ &= a_0 + 2a_1X + 3a_2X^2 + \cdots + (n+1)a_nX^n. \end{aligned}$$

Set this equal to the zero polynomial:

$$a_0 = 0, \quad 2a_1 = 0, \quad 3a_2 = 0, \quad \dots, \quad (n+1)a_n = 0.$$

This implies:

$$a_0 = a_1 = a_2 = \cdots = a_n = 0.$$

Therefore, the only solution is the zero polynomial:

$$\ker f = \{0\}.$$

3. Image of f :

Let us compute f on each basis element:

$$f(1) = 1 + X \cdot 0 = 1, \quad f(X) = X + X \cdot 1 = 2X,$$

$$f(X^2) = X^2 + X \cdot 2X = 3X^2, \quad f(X^3) = X^3 + X \cdot 3X^2 = 4X^3.$$

\vdots

$$f(X^n) = X^n + X \cdot nX^{n-1} = (n+1)X^n.$$

So:

$$f(1) = 1, \quad f(X) = 2X, \quad f(X^2) = 3X^2, \quad \dots, \quad f(X^n) = (n+1)X^n.$$

We see that f sends each basis monomial X^k to a scalar multiple of X^k .

Hence:

$$\text{Im}(f) = \text{Span}\{1, X, X^2, \dots, X^n\} = \mathbb{R}_n[X].$$

Since this is true for arbitrary n , we conclude:

$$\text{Im}(f) = \mathbb{R}[X].$$

Thus, the image of f is the entire space of real polynomials.

4. Injectivity and Surjectivity:

- f is injective since $\ker f = \{0\}$. - f is surjective since $\text{Im}(f) = \mathbb{R}[X]$.

Therefore, f is an automorphism of $\mathbb{R}[X]$ (a bijective linear operator).

Solution 2.3

Let f be a function defined from \mathbb{R}^3 to \mathbb{R}^3 by:

$$f(x, y, z) = (2x + y, y - z, x - y).$$

1. Linearity of f :

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in \mathbb{R}^3$, and $\lambda \in \mathbb{R}$.

Additivity:

$$\begin{aligned} f(u+v) &= f(x_1+x_2, y_1+y_2, z_1+z_2) \\ &= (2(x_1+x_2) + (y_1+y_2), (y_1+y_2) - (z_1+z_2), (x_1+x_2) - (y_1+y_2)) \\ &= (2x_1+2x_2+y_1+y_2, y_1-z_1+y_2-z_2, x_1-y_1+x_2-y_2) \\ &= f(u) + f(v). \end{aligned}$$

Homogeneity:

$$f(\lambda u) = f(\lambda x_1, \lambda y_1, \lambda z_1) = (2\lambda x_1 + \lambda y_1, \lambda y_1 - \lambda z_1, \lambda x_1 - \lambda y_1) = \lambda f(u).$$

Therefore, f is linear.

2. Kernel and image of f :

We solve:

$$f(x, y, z) = (2x + y, y - z, x - y) = (0, 0, 0) \Rightarrow \begin{cases} 2x + y = 0, \\ y - z = 0, \\ x - y = 0. \end{cases}$$

From the third equation, $x = y$. Substituting into the first gives $2y + y = 3y = 0 \Rightarrow y = 0 \Rightarrow x = 0$. From the second, $z = y = 0 \Rightarrow z = 0$.

Thus, the only solution is $(0, 0, 0)$, hence:

$$\ker f = \{0\}, \quad \dim(\ker f) = 0.$$

By the Rank–Nullity Theorem:

$$\dim(\operatorname{Im} f) = 3 - 0 = 3 \Rightarrow \operatorname{Im} f = \mathbb{R}^3.$$

Therefore, f is bijective (injective and surjective).

3. Compute $f \circ f$:

Let $f(x, y, z) = (u, v, w) = (2x + y, y - z, x - y)$.

We now compute $f(u, v, w)$:

$$f(u, v, w) = (2u + v, v - w, u - v).$$

Substitute $u = 2x + y$, $v = y - z$, $w = x - y$.

First component:

$$2u + v = 2(2x + y) + (y - z) = 4x + 2y + y - z = 4x + 3y - z.$$

Second component:

$$v - w = (y - z) - (x - y) = y - z - x + y = -x + 2y - z.$$

Third component:

$$u - v = (2x + y) - (y - z) = 2x + y - y + z = 2x + z.$$

Therefore:

$$(f \circ f)(x, y, z) = (4x + 3y - z, -x + 2y - z, 2x + z).$$

Solution 2.4

Let $B = \{e_1, e_2, e_3\}$ be the canonical basis of \mathbb{R}^3 , and let f be the endomorphism of \mathbb{R}^3 defined by:

$$f(e_1) = -2e_1 + 2e_3, \quad f(e_2) = 3e_2, \quad f(e_3) = -4e_1 + 4e_3.$$

1. **Compute $f(x, y, z)$:**

Let $(x, y, z) = xe_1 + ye_2 + ze_3$. By linearity:

$$f(x, y, z) = xf(e_1) + yf(e_2) + zf(e_3) = x(-2e_1 + 2e_3) + y(3e_2) + z(-4e_1 + 4e_3).$$

Compute:

$$\begin{aligned} f(x, y, z) &= (-2x - 4z)e_1 + 3ye_2 + (2x + 4z)e_3 \\ &\Rightarrow f(x, y, z) = (-2x - 4z, 3y, 2x + 4z). \end{aligned}$$

2. **Find a basis of $\ker f$:**

We solve:

$$f(x, y, z) = (0, 0, 0) \Rightarrow \begin{cases} -2x - 4z = 0, \\ 3y = 0, \\ 2x + 4z = 0. \end{cases}$$

From the second equation, $y = 0$. From the first, $-2x = 4z \Rightarrow x = -2z$.

Thus:

$$(x, y, z) = (-2z, 0, z) = z(-2, 0, 1) \Rightarrow \ker f = \text{Span}\{(-2, 0, 1)\}.$$

Hence, a basis of the kernel is $\{(-2, 0, 1)\}$, and $\dim(\ker f) = 1$.

3. Injectivity and surjectivity:

Since $\ker f \neq \{0\}$, the map f is not injective. Also, $\dim(\ker f) = 1 \Rightarrow \dim(\operatorname{Im} f) = 2$, so f cannot be surjective (since $\dim(\mathbb{R}^3) = 3$).

Therefore:

f is not injective and not surjective.

4. Basis of $\operatorname{Im} f$ and rank:

We use the images of the basis vectors:

$$f(e_1) = (-2, 0, 2), \quad f(e_2) = (0, 3, 0), \quad f(e_3) = (-4, 0, 4).$$

Note that:

$$f(e_3) = 2f(e_1) \Rightarrow \text{dependent.}$$

So:

$$\operatorname{Im} f = \operatorname{Span}\{f(e_1), f(e_2)\}.$$

A basis is $\{(-2, 0, 2), (0, 3, 0)\}$, and $\operatorname{rank}(f) = 2$.

5. Direct sum decomposition:

We already have:

$$\dim(\ker f) = 1, \quad \dim(\operatorname{Im} f) = 2 \Rightarrow \dim(\ker f) + \dim(\operatorname{Im} f) = 3 = \dim(\mathbb{R}^3).$$

Since $\ker f \cap \operatorname{Im} f = \{0\}$, we conclude that:

$$\mathbb{R}^3 = \ker f \oplus \operatorname{Im} f.$$

Solution 2.5

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear map such that:

$$(1, 2, 0) \in \ker(f), \quad f(0, 0, 1) = (1, 0), \quad f(0, t, 0) = (t, t) \quad \forall t \in \mathbb{R}.$$

1. Determine the expression of $f(x, y, z)$:

We are given:

$$f(0, 0, 1) = (1, 0), \quad f(0, t, 0) = (t, t), \quad f(1, 2, 0) = (0, 0).$$

Let us find the image of the standard basis vectors:

$$f(0, 1, 0) = (1, 1) \quad (\text{from the second property}),$$

$$f(0, 0, 1) = (1, 0) \quad (\text{given}).$$

Now, by linearity:

$$f(x, y, z) = xf(1, 0, 0) + yf(0, 1, 0) + zf(0, 0, 1).$$

Let us find $f(1, 0, 0)$ using the fact that $f(1, 2, 0) = (0, 0)$:

$$\begin{aligned} f(1, 2, 0) &= f(1, 0, 0) + 2f(0, 1, 0) \\ &= f(1, 0, 0) + 2(1, 1) \\ &= (0, 0) \Rightarrow f(1, 0, 0) = (-2, -2). \end{aligned}$$

Therefore:

$$f(x, y, z) = x(-2, -2) + y(1, 1) + z(1, 0) = (-2x + y + z, -2x + y).$$

2. Basis and dimensions of $\ker(f)$ and $\text{Im}(f)$:

We solve:

$$\begin{aligned} f(x, y, z) = (0, 0) &\Rightarrow \begin{cases} -2x + y + z = 0, \\ -2x + y = 0. \end{cases} \\ &\Rightarrow z = 0, \quad y = 2x. \end{aligned}$$

Thus:

$$\ker(f) = \{(x, 2x, 0) \mid x \in \mathbb{R}\}.$$

A basis of $\ker(f)$ is $\{(1, 2, 0)\}$, and:

$$\dim(\ker f) = 1.$$

To find $\text{Im}(f)$, consider the images of the basis vectors:

$$f(1, 0, 0) = (-2, -2), \quad f(0, 1, 0) = (1, 1), \quad f(0, 0, 1) = (1, 0).$$

Note that the vectors $(1, 1)$ and $(1, 0)$ are linearly independent; hence:

$$\dim(\text{Im}(f)) = 2 \Rightarrow \text{Im}(f) = \mathbb{R}^2.$$

A basis is $\{(1, 1), (1, 0)\}$.

3. Find all vectors whose image is $(0, 1)$. Is it a subspace?

We solve:

$$f(x, y, z) = (0, 1) \Rightarrow \begin{cases} -2x + y + z = 0, \\ -2x + y = 1. \end{cases}$$

Subtract the two equations:

$$z = -1, \quad y = 1 + 2x.$$

So, the set of solutions is:

$$\{(x, 1 + 2x, -1) \mid x \in \mathbb{R}\} = (0, 1, -1) + x(1, 2, 0).$$

This set is not a vector subspace since it does not contain the zero vector.

Solution 2.6

We are given vector spaces E, F, G over a field \mathbb{K} , and two linear maps:

$$f \in \mathcal{L}(E, F), \quad g \in \mathcal{L}(F, G).$$

We are to prove:

$$g \circ f = 0 \iff \text{Im}(f) \subset \ker(g).$$

(\Rightarrow) Suppose $g \circ f = 0$.

This means that for every $x \in E$, we have:

$$g(f(x)) = 0 \Rightarrow f(x) \in \ker(g) \Rightarrow \text{Im}(f) \subset \ker(g).$$

(\Leftarrow) Conversely, suppose $\text{Im}(f) \subset \ker(g)$.

Then, for every $x \in E$, since $f(x) \in \text{Im}(f) \subset \ker(g)$, we have:

$$g(f(x)) = 0 \Rightarrow g \circ f = 0.$$

Conclusion: Both directions are proven; hence:

$$g \circ f = 0 \iff \text{Im}(f) \subset \ker(g).$$

Solution 2.7

Let E be a real vector space of dimension 3, and $f \in \mathcal{L}(E)$ such that $f^3 = 0$, $f^2 \neq 0$, and $f^2(x_0) \neq 0$.

1. Show that $B = \{x_0, f(x_0), f^2(x_0)\}$ is a basis of E :

Let us show that the set B is linearly independent. Suppose:

$$\lambda_0 x_0 + \lambda_1 f(x_0) + \lambda_2 f^2(x_0) = 0. \quad (1)$$

Apply f to both sides:

$$\lambda_0 f(x_0) + \lambda_1 f^2(x_0) + \lambda_2 f^3(x_0) = 0 \Rightarrow \lambda_0 f(x_0) + \lambda_1 f^2(x_0) = 0. \quad (2)$$

Apply f again:

$$\lambda_0 f^2(x_0) + \lambda_1 f^3(x_0) = \lambda_0 f^2(x_0) = 0. \quad (3)$$

Since $f^2(x_0) \neq 0$, we get $\lambda_0 = 0$. Then, from (2), $\lambda_1 f^2(x_0) = 0 \Rightarrow \lambda_1 = 0$. Back to (1), $\lambda_2 f^2(x_0) = 0 \Rightarrow \lambda_2 = 0$. So, B is linearly independent; and since $\dim(E) = 3$, it forms a basis.

2. Let $g \in \mathcal{L}(E)$ such that $g \circ f = f \circ g$.

(a) Show that there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that:

$$g(x_0) = \alpha x_0 + \beta f(x_0) + \gamma f^2(x_0).$$

Since $\{x_0, f(x_0), f^2(x_0)\}$ is a basis of E , then $g(x_0) \in E$ can be written uniquely as:

$$g(x_0) = \alpha x_0 + \beta f(x_0) + \gamma f^2(x_0),$$

for some scalars $\alpha, \beta, \gamma \in \mathbb{R}$.

(b) Show that:

$$g = \alpha \text{Id}_E + \beta f + \gamma f^2.$$

Let us show that for all $v \in E$, we have:

$$g(v) = \alpha v + \beta f(v) + \gamma f^2(v).$$

Since $B = \{x_0, f(x_0), f^2(x_0)\}$ is a basis, it is enough to verify this formula on each element of B .

First:

$$g(x_0) = \alpha x_0 + \beta f(x_0) + \gamma f^2(x_0) \quad (\text{by assumption}).$$

Second: Apply $f \circ g = g \circ f$ on x_0 :

$$\begin{aligned} f(g(x_0)) &= f(\alpha x_0 + \beta f(x_0) + \gamma f^2(x_0)) \\ &= \alpha f(x_0) + \beta f^2(x_0) + \gamma f^3(x_0) = \alpha f(x_0) + \beta f^2(x_0), \end{aligned}$$

and

$$g(f(x_0)) = \alpha f(x_0) + \beta f^2(x_0) + \gamma f^3(x_0) = \alpha f(x_0) + \beta f^2(x_0).$$

So, the formula holds on $f(x_0)$.

Similarly:

$$g(f^2(x_0)) = \alpha f^2(x_0).$$

Therefore:

$$g(v) = \alpha v + \beta f(v) + \gamma f^2(v), \quad \forall v \in E \Rightarrow g = \alpha \text{Id}_E + \beta f + \gamma f^2.$$

(c) Deduce the set of endomorphisms commuting with f :

From the previous result, any endomorphism $g \in \mathcal{L}(E)$ such that $g \circ f = f \circ g$ must be of the form:

$$g = \alpha \text{Id}_E + \beta f + \gamma f^2.$$

Hence, the set of all such endomorphisms is:

$$\{\alpha \text{Id}_E + \beta f + \gamma f^2 \mid \alpha, \beta, \gamma \in \mathbb{R}\},$$

which forms a 3-dimensional subspace of $\mathcal{L}(E)$.