

Chapter 2

Linear Mappings

Linear Mappings

Linear mappings are the cornerstone of linear algebra, providing a systematic framework to study structure-preserving transformations between vector spaces. This chapter explores their fundamental properties, classifications, and applications across mathematics.

A linear map $f : E \rightarrow F$ is characterized by its ability to commute with vector space operations:

$$f(\alpha u + v) = \alpha f(u) + f(v) \quad \forall u, v \in E, \alpha \in \mathbb{K}.$$

We will examine key examples—from differentiation and integration to matrix transformations—and establish tools to analyze their behavior through kernels, images, and the rank-nullity theorem.

Key Objectives

- Characterize linearity via additivity and homogeneity.
- Compute kernels and images, and relate their dimensions.
- Classify maps by injectivity, surjectivity, and isomorphisms.
- Construct linear maps from basis assignments.
- Apply rank inequalities to solve dimension constraints.

Notation 2.1

- $\mathcal{L}(E, F)$: space of linear maps $E \rightarrow F$;
- $\ker f$: kernel (null space) of f ;
- $\text{Im } f$: image of f ;
- $\text{rg}(f)$: rank of f .

2.1 Definitions**Definition 2.1: Linear Mapping**

Let E, F be two vector spaces over a field \mathbb{K} . A linear mapping (or linear transformation) $f : E \rightarrow F$ is a mapping that satisfies the following two properties:

1. **Additivity:** For any vectors $u, v \in E$,

$$f(u + v) = f(u) + f(v).$$

①

2. **Homogeneity:** For any vector $u \in E$ and any scalar $c \in \mathbb{K}$,

$$f(cu) = cf(u).$$

②

Let's look at some examples of linear maps.

Example 2.1: Linear Maps

1. **Zero linear map:** For vector spaces E and F over the same field \mathbb{K} , the zero linear map is defined by

$$0 : E \rightarrow F, \quad 0(x) = 0_F \text{ for all } x \in E,$$

where 0_F denotes the zero vector in F .

2. **Identity map:** On a vector space E , the identity map is given by

$$\text{Id}_E : E \rightarrow E, \quad \text{Id}_E(x) = x \text{ for all } x \in E.$$

This map is linear and is sometimes denoted simply by I when the space is clear from context.

$\mathbb{R}[x]$

3. **Differentiation:** Let $P(\mathbb{R})$ be the space of real polynomials. Define

~~$D \in \mathcal{L}(P(\mathbb{R}))$ by $D(p) = p'$.~~

$D: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$
 $D(p+q) = p' + q'$
 $D(p) = p'$

Differentiation is a linear map.

4. **Integration:** Define

$$T \in \mathcal{L}(P(\mathbb{R}), \mathbb{R}) \text{ by } T(p) = \int_0^1 p(x) dx.$$

Integration is also a linear map.

5. **Matrix-type linear map:** Define

$$T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2) \text{ by } T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z).$$

This corresponds to multiplication by the matrix

~~$$\begin{pmatrix} 2 & -1 & 3 \\ 7 & 5 & -6 \end{pmatrix}$$~~

Theorem 2.1: Linear Map Characterization

Let E and F be two \mathbb{K} -vector spaces, and let $f : E \rightarrow F$ be a function. Then f is a linear map if and only if

$$\forall x, y \in E, \forall \lambda \in \mathbb{K}, \quad f(\lambda x + y) = \lambda f(x) + f(y).$$

Remark 2.1: Linear Map Terminology

1. If a linear map f is injective, we say that f is a monomorphism.
2. If a linear map f is surjective, we say that f is an epimorphism.
3. If a linear map f is bijective, we say that f is an isomorphism of vector spaces, and that E and F are isomorphic.

- $f: E \rightarrow E$
4. A bijective endomorphism is called an *automorphism*.
 5. The set of all linear mappings from E to F is denoted by $\mathcal{L}(E, F)$.
 6. The set of linear mappings from E to itself is denoted by $\mathcal{L}(E)$; in other words, $\mathcal{L}(E) = \mathcal{L}(E, E)$.

Proposition 2.1: Vector Space of Linear Maps

Let E and F be vector spaces over \mathbb{K} . The set $\mathcal{L}(E, F)$ of all linear maps from E to F , equipped with:

- Addition: $(f + g)(x) = f(x) + g(x)$;
- Scalar multiplication: $(\lambda f)(x) = \lambda f(x)$,

forms a vector space over \mathbb{K} .

$f + g$
 $\lambda \cdot f$
 $(\mathcal{L}(E, F), +, \cdot) \vee \mathbb{K}$ vector space

Proposition 2.2: Composition of Linear Maps

Let E, F, G be vector spaces over \mathbb{K} . If $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$, then their composition $g \circ f$ belongs to $\mathcal{L}(E, G)$.

Theorem 2.2

Let f be a linear map. Then:

1. $f(0_E) = 0_F$;
2. If A is a subspace of E , then the restriction $f|_A$ is a linear map on A ;
3. For all $x \in E$, $f(-x) = -f(x)$;
4. For any scalars $\alpha_1, \dots, \alpha_n$ and vectors $x_1, \dots, x_n \in E$,

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) = \sum_{i=1}^n \alpha_i f(x_i).$$

Theorem 2.3: Existence and Uniqueness of a Linear Map from Basis Assignment

Let E and F be two vector spaces over the same field \mathbb{K} . Suppose E is finite-dimensional with dimension n , and let (e_1, \dots, e_n) be a basis of E . Then for any choice of n vectors (v_1, \dots, v_n) in F , there exists a unique linear map $f : E \rightarrow F$ such that, for all $i = 1, \dots, n$,

$$f(e_i) = v_i.$$

This theorem makes no assumption about the dimension of the target vector space F .

Proof 2.1

Uniqueness. Suppose there exists a linear map $f : E \rightarrow F$ such that $f(e_i) = v_i$ for all $i = 1, \dots, n$. For any $x \in E$, there exist unique scalars x_1, \dots, x_n such that

$$x = \sum_{i=1}^n x_i e_i.$$

Since f is linear, we have

$$f(x) = f\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i f(e_i) = \sum_{i=1}^n x_i v_i. \quad (*)$$

Therefore, if such f exists, it must be unique.

Existence. Define the map $f : E \rightarrow F$ by equation (*) as

$$f\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i v_i.$$

We show that this map is linear and satisfies $f(e_i) = v_i$.

Let (x_1, \dots, x_n) (respectively, (y_1, \dots, y_n)) be the coordinates of x (respectively,

y) in the basis (e_1, \dots, e_n) . Then, for any scalars $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned} f(\alpha x + \beta y) &= f\left(\sum_{i=1}^n (\alpha x_i + \beta y_i) e_i\right) \\ &= \sum_{i=1}^n (\alpha x_i + \beta y_i) v_i \\ &= \alpha \sum_{i=1}^n x_i v_i + \beta \sum_{i=1}^n y_i v_i \\ &= \alpha f(x) + \beta f(y). \end{aligned}$$

Finally, since the coordinates of e_i are $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i -th position, we have

$$f(e_i) = 1 \cdot v_i = v_i.$$

This completes the proof of the theorem.

Example 2.2

There exists a unique linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}[X]$ such that

$$f(e_i) = (X + 1)^i, \quad \text{for } i = 1, \dots, n,$$

where (e_1, \dots, e_n) is the standard basis of \mathbb{R}^n .

For a vector $x = (x_1, \dots, x_n)$, we have

$$f(x_1, \dots, x_n) = f\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i f(e_i) = \sum_{i=1}^n x_i (X + 1)^i.$$

Theorem 2.4: Characterization of Injectivity and Surjectivity via Basis Image

Let $f : E \rightarrow F$ be a linear map, where E has a basis $(e_i)_{i \in I}$. Then:

e_i :

1. f is surjective if and only if $F = \text{span}\{f(e_i)\}_{i \in I}$;
2. f is injective if and only if $(f(e_i))_{i \in I}$ is linearly independent;
3. f is bijective if and only if $(f(e_i))_{i \in I}$ is a basis of F .

Example 2.3: Surjective Map

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = x + y$. Consider the basis $(e_1, e_2) = ((1, 0), (0, 1))$ of \mathbb{R}^2 :

- $f(e_1) = 1, f(e_2) = 1$;
- $\text{Span}\{f(e_1), f(e_2)\} = \text{Vect}\{1\} = \mathbb{R}$.

Thus, f is surjective.

Example 2.4: Injective Map

Consider $g : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $g(x, y) = (x, y, 0)$:

- $g(e_1) = (1, 0, 0), g(e_2) = (0, 1, 0)$;
- The set $\{g(e_1), g(e_2)\}$ is linearly independent.

Hence, g is injective.

$e_1 = (1, 0), e_2 = (0, 1)$
 $\mathbb{R}^3 \neq \text{Span}\{g(e_1), g(e_2)\}$

Example 2.5: Bijective Map

Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $h(x, y) = (y, x)$: $e_1 = (1, 0), e_2 = (0, 1)$

- $h(e_1) = e_2, h(e_2) = e_1$;
- The set $\{h(e_1), h(e_2)\} = \{e_2, e_1\}$ forms a basis of \mathbb{R}^2 .

Therefore, h is bijective by Theorem 2.9(iii).

Theorem 2.5: Equivalence of Injectivity, Surjectivity, and Bijectivity

Let E and F be finite-dimensional \mathbb{K} -vector spaces of the same dimension, and let $f : E \rightarrow F$ be a linear transformation. Then:

$$f \text{ is bijective} \iff f \text{ is injective} \iff f \text{ is surjective.}$$

2.2 Image and Kernel of a Linear map

Definition 2.2: Kernel of a Linear Map

Let $f : E \rightarrow F$ be a linear map. The **kernel** of f , denoted by $\ker f$, is the set defined by

$$\ker f = \{u \in E \mid f(u) = 0_F\}.$$

It is sometimes also denoted by $f^{-1}(0)$.

Example 2.6

Determine the kernel of the linear map f_1 , defined by

$$f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto x + 2y.$$

We are looking for

$$\ker f_1 = \{(x, y) \in \mathbb{R}^2 \mid f_1(x, y) = 0\} = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 0\}.$$

Solving the equation $x + 2y = 0$, we get

$$x = -2y.$$

Thus, every element in the kernel has the form

$$(x, y) = (-2y, y) = y(-2, 1), \quad y \in \mathbb{R}.$$

Therefore,

$$\ker f_1 = \text{Span}\{(-2, 1)\}.$$

This means that the kernel is a vector subspace of \mathbb{R}^2 of dimension 1, and a basis of $\ker f_1$ is $\{(-2, 1)\}$.

Proposition 2.3: Preimage of a Subspace

Let $f : E \rightarrow F$ be a linear map. Then:

1. More generally, if $B \subset F$ is a subspace, then the preimage

$$f : E \rightarrow F$$

$$f^{-1}(B) = \{x \in E \mid f(x) \in B\}$$

is a subspace of E . In particular, since $\{0\} \subset F$ is a subspace, it follows that

$$\ker f = f^{-1}(\{0\})$$

$$f^{-1}(\{0\}) = \ker f$$

is a subspace of E .

2. The map f is injective if and only if

$$\ker f = \{0\}.$$

Example 2.7: Kernel as a Special Case

For the linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$f(x, y, z) = (x + y, y - z),$$

the kernel is

$$\ker f = \{(x, y, z) \mid x + y = 0, y - z = 0\} = \{(-y, y, y) \mid y \in \mathbb{R}\}.$$

This line through the origin shows that $\ker f$ is indeed a subspace, as it is the preimage of $\{0\}$.

Example 2.8: Injectivity Criterion

Define $g : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by

$$g(x, y) = (x, 2y, x + y).$$

- $\ker g = \{(0, 0)\}$, since $g(x, y) = (0, 0, 0)$ implies $x = y = 0$;
- By Proposition 2.12(ii), g is injective;
- Visually, no two distinct input vectors map to the same output.

Contrast with $h : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$h(x, y) = (x + y, x + y, 0),$$

where

$$\ker h = \{(x, -x) \mid x \in \mathbb{R}\} \neq \{(0, 0)\}.$$

Thus, h is not injective, showing that the kernel criterion works.

Definition 2.3: Image of a Linear Map

Let $f : E \rightarrow F$ be a linear map between \mathbb{K} -vector spaces. The **image** of f , denoted $\text{Im}(f)$, is the subset of F defined by

$$\text{Im}(f) := \{f(x) \mid x \in E\} = f(E).$$

Example 2.9: (Finding the Image of a Linear Transformation)

Consider the linear transformation

$$f_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

defined by

$$f_2(x, y, z) = (-x + y, x - z, y).$$

Image Calculation. The image of f_2 is

$$\begin{aligned} \text{Im } f_2 &= \{f_2(x, y, z) \mid (x, y, z) \in \mathbb{R}^3\} \\ &= \{(-x + y, x - z, y) \mid (x, y, z) \in \mathbb{R}^3\} \\ &= \{x(-1, 1, 0) + y(1, 0, 1) + z(0, -1, 0) \mid (x, y, z) \in \mathbb{R}^3\}. \end{aligned}$$

Thus, $\text{Im } f_2$ is a subspace of \mathbb{R}^3 generated by the vectors

$$\{(-1, 1, 0), (1, 0, 1), (0, -1, 0)\}.$$

Basis and Dimension. It is easy to show that this family of vectors is linearly

independent and forms a basis for \mathbb{R}^3 . Therefore,

$$\dim \text{Im } f_2 = 3 \text{ and } \text{Im } f_2 = \mathbb{R}^3$$

Theorem 2.6: (Image of a Vector Subspace under a Linear Map)

Let f be a linear map from E to F .

1. If A is a vector subspace of E , then $f(A)$ is a vector subspace of F . In particular, $\text{Im}(f) = f(E)$ is a vector subspace of F .
2. f is surjective if and only if $\text{Im}(f) = F$.

Example 2.10

For $n \geq 1$, consider the linear map

$$f : \mathbb{R}_n[X] \rightarrow \mathbb{R}_{n-1}[X], \quad P \mapsto P',$$

where P' denotes the derivative of P .

The image of f is

$$\begin{aligned} \text{Im}(f) &= \{P' \mid P \in \mathbb{R}_n[X]\} \\ &= \{a_1 + 2a_2X + \dots + na_nX^{n-1} \mid a_1, \dots, a_n \in \mathbb{R}\} \\ &= \mathbb{R}_{n-1}[X]. \end{aligned}$$

which shows that f is surjective.

Moreover,

$$\ker(f) = \{P \in \mathbb{R}_n[X] \mid P' = 0\} = \mathbb{R}_0[X],$$

the space of constant polynomials. Thus, for $n \geq 1$, f is not injective.

Theorem 2.7: (Image of a Span under a Linear Map)

Let f be a linear map from E to F . For any subset X of E :

$$f(\text{Span}(X)) = \text{Span}(f(X)).$$

$$\begin{aligned} f(1, 4) &= (3, 2, 4) \\ f(1, 2) &= (3, 2) \end{aligned}$$

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\begin{aligned} f(\text{Span}\{(1, 2)\}) &= \text{Span}(f(1, 2)) \\ &= \text{Span}(3, 2) \end{aligned}$$

In particular, if E has a basis $(e_i)_{i \in I}$:

$$\text{Im}(f) = \text{Span}\{f(e_i) \mid i \in I\}.$$

Proof 2.2

(1) To prove the equality $f(\text{Span}(X)) = \text{Span}(f(X))$:

(i) Let $y \in f(\text{Span}(X))$. Then there exist $x_1, \dots, x_n \in X$ and scalars $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ such that

$$y = f\left(\sum_{i=1}^n \lambda_i x_i\right).$$

By linearity of f ,

$$y = \sum_{i=1}^n \lambda_i f(x_i) \in \text{Span}(f(X)).$$

Hence, $f(\text{Span}(X)) \subseteq \text{Span}(f(X))$.

(ii) Conversely, let $y \in \text{Span}(f(X))$. Then there exist $x_1, \dots, x_n \in X$ and scalars $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ such that

$$y = \sum_{i=1}^n \lambda_i f(x_i) = f\left(\sum_{i=1}^n \lambda_i x_i\right).$$

Since $\sum_{i=1}^n \lambda_i x_i \in \text{Span}(X)$, we have $y \in f(\text{Span}(X))$. Thus, $\text{Span}(f(X)) \subseteq f(\text{Span}(X))$.

Combining (i) and (ii), we get $f(\text{Span}(X)) = \text{Span}(f(X))$.

(2) For the particular case $X = \{e_i\}_{i \in I}$, a basis of E , we have

$$\text{Span}(\{e_i\}_{i \in I}) = E, \quad \text{hence} \quad f(E) = \text{Im}(f) = \text{Span}\{f(e_i) \mid i \in I\}.$$

Example 2.11

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by $f(x, y, z) = (2x - y, y + z)$.

Take $X = \{(1, 0, 0), (0, 1, 0)\}$.

1. Computing both sides:

$$\begin{aligned} f(\text{Span}(X)) &= f(\{a(1, 0, 0) + b(0, 1, 0) \mid a, b \in \mathbb{R}\}) \\ &= f(\{(a, b, 0) \mid a, b \in \mathbb{R}\}) \\ &= \mathbb{R}^2. \end{aligned}$$

$$\text{Span}(f(X)) = \text{Span}\{f(1, 0, 0), f(0, 1, 0)\} = \text{Span}\{(2, -1), (0, 1)\} = \mathbb{R}^2.$$

2. Using the standard basis $\{e_1, e_2, e_3\}$:

$$\text{Im}(f) = \text{Span}\{f(e_1), f(e_2), f(e_3)\} = \text{Span}\{(2, 0), (-1, 1), (0, 1)\} = \mathbb{R}^2.$$

2.3 Rank-Nullity Theorem**Definition 2.4: (Rank of a Linear Map)**

Let f be a linear map from E to F . We say that f has **finite rank** if $\text{Im}(f)$ is finite-dimensional, and **infinite rank** otherwise. When f has finite rank, the **rank** of f , denoted by $\text{rg}(f)$, is defined as

$$\text{rg}(f) = \dim(\text{Im}(f)).$$

Theorem 2.8: (Rank-Nullity Theorem)

Let $f : E \rightarrow F$ be a linear map between \mathbb{K} -vector spaces. If E is finite-dimensional, then

$$\dim E = \dim(\ker f) + \text{rg}(f).$$

$$= \dim(\ker f) + \dim(\text{Im } f)$$

Example 2.12

Consider the differentiation map:

$$D : \mathbb{R}_3[X] \rightarrow \mathbb{R}_2[X], \quad P \mapsto P'.$$

- $\dim \mathbb{R}_3[X] = 4$ (basis: $\{1, X, X^2, X^3\}$);
- $\ker D = \{P \mid P' = 0\} = \mathbb{R}_0[X]$, with $\dim \ker D = 1$;
- $\text{Im}(D) = \mathbb{R}_2[X]$, with $\text{rg}(D) = 3$.

Verification: $4 = 1 + 3$ confirms the theorem.

Theorem 2.9: (Rank Inequalities)

Let $f : E \rightarrow F$ be a linear map between \mathbb{K} -vector spaces.

1. If F is finite-dimensional, then f has finite rank and

$$\text{Im } f \text{ ssv of } F \\ \dim \text{Im } f \leq \dim F$$

$$\text{rg}(f) \leq \dim F.$$

Moreover, f is surjective if and only if $\text{rg}(f) = \dim F$.

2. If E is finite-dimensional, then f has finite rank and

$$\text{rg}(f) \leq \dim E.$$

Moreover, f is injective if and only if $\text{rg}(f) = \dim E$.

$$\dim E = \dim \ker f + \text{rg } f \\ \text{rg } f \leq \dim E$$

Proof 2.3

1. Since $\text{Im}(f) \subset F$ is a subspace, if $\dim F < \infty$, then

$$\text{rg}(f) = \dim(\text{Im}(f)) \leq \dim F.$$

Surjectivity is equivalent to $\text{Im}(f) = F$, which implies equality of dimensions.

2. When $\dim E < \infty$, the Rank–Nullity Theorem gives

$$\operatorname{rg}(f) = \dim E - \dim(\ker f) \leq \dim E.$$

Injectivity ($\ker f = \{0\}$) occurs precisely when $\dim(\ker f) = 0$, yielding $\operatorname{rg}(f) = \dim E$.

Example 2.13

Consider the projection $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$\pi(x, y, z) = (x, y).$$

- $\dim F = 2$, $\operatorname{rg}(\pi) = 2 \leq 2$ (surjective);
- $\dim E = 3$, $\operatorname{rg}(\pi) = 2 < 3$ (not injective).