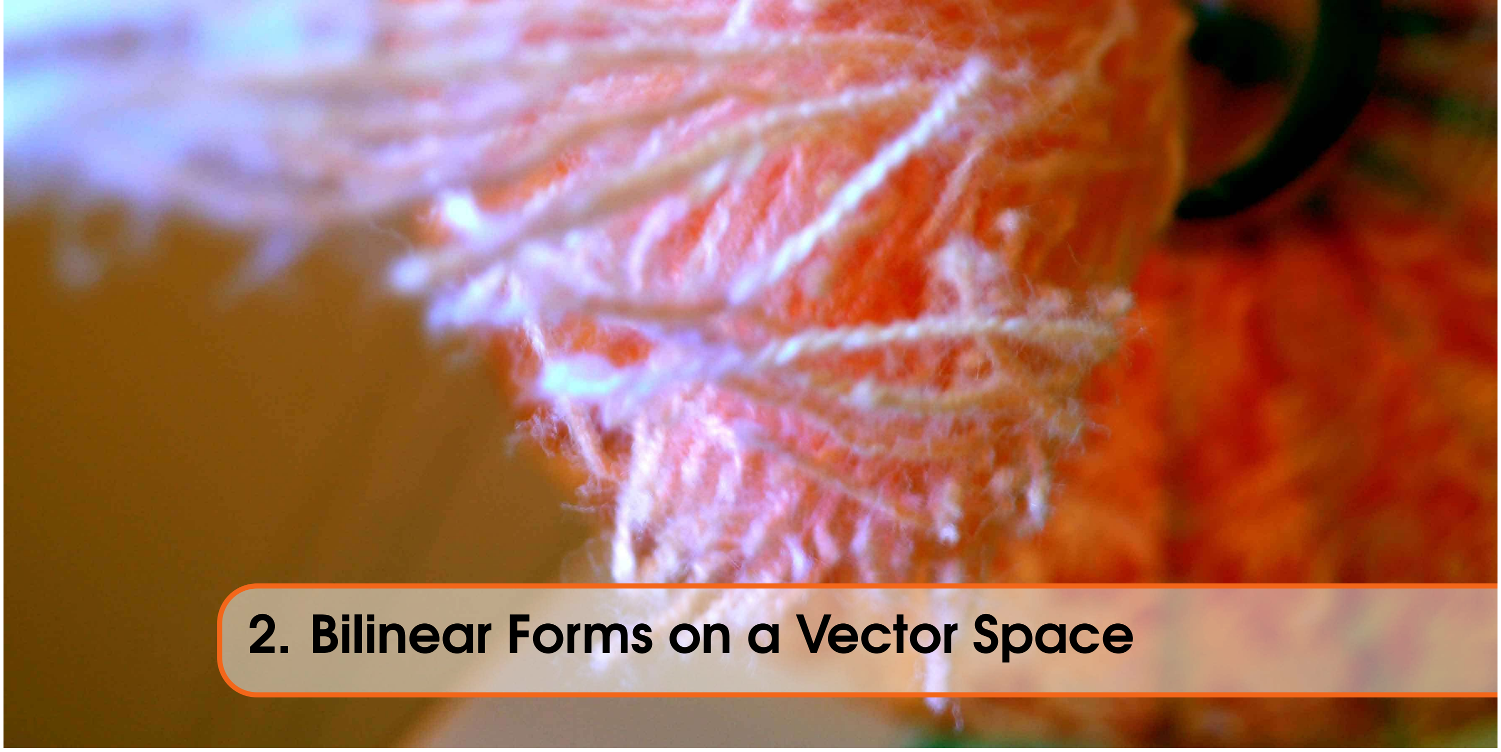

2. Bilinear Forms on a Vector Space

Throughout this chapter, E, F, and G are vector spaces over the same field $\mathbb{K}$.

2.1 Bilinear Forms

Definition 2.1.1 A mapping $f : E \times F \longrightarrow G$ is said to be **bilinear** if:

$$\begin{aligned} f(x_1+x_2,y) &= f(x_1,y)+f(x_2,y) \\ f(\alpha x,y) &= \alpha f(x,y) \\ f(x,y_1+y_2) &= f(x,y_1)+f(x,y_2) \\ f(x,\alpha y) &= \alpha f(x,y) \end{aligned}$$

for all $x,x_1,x_2 \in E$, $y,y_1,y_2 \in F$ and $\alpha \in \mathbb{K}$.

If $G = \mathbb{K}$, f is called a **bilinear form**.

Definition 2.1.2 A mapping $f : E \times F \longrightarrow G$ is said to be bilinear if:

i) For every $x \in E$ (with x fixed), the mapping

$$\begin{aligned} f_x : \quad F &\longrightarrow G \\ y &\longmapsto f(x,y) \end{aligned}$$

is a linear form on F.

ii) For every $y \in F$ (with y fixed), the mapping

$$\begin{aligned} f_y : \quad E &\longrightarrow G \\ x &\longmapsto f(x,y) \end{aligned}$$

is a linear form on E.

■ **Example 2.1** 1. The mapping

$$\begin{aligned} f_0 : \quad E \times F &\longrightarrow \mathbb{K} \\ (x,y) &\longmapsto 0_{\mathbb{K}} \end{aligned}$$

is a bilinear form.

2. The mapping
$$\begin{array}{rrl} g: & \mathbb{R}\times\mathbb{R} & \longrightarrow \mathbb{R} \\ & (x,y) & \longmapsto xy \end{array}$$
is a bilinear form.
3. Let $\mathscr{C}([0,1],\mathbb{R})=\{f:[0,1]\to\mathbb{R}:\ f \text{ continuous }\}$. The mapping
$$\begin{array}{rrl} \varphi: & \mathscr{C}([0,1],\mathbb{R})\times\mathscr{C}([0,1],\mathbb{R}) & \longrightarrow \mathbb{R} \\ & (f,g) & \longmapsto \displaystyle\int_0^1 f(t)g(t)dt \end{array}$$
is a bilinear form.

■

2.1.1 The Vector Space $\mathscr{L}(E\times F,\mathbb{K})$

Proposition 2.1.1 The set of all bilinear forms from $E\times F$ into $\mathbb{K}$, denoted $\mathscr{L}(E\times F,\mathbb{K})$, is a vector space over $\mathbb{K}$.

Proof. Let $\alpha,\beta\in\mathbb{K}$ et $f,g\in\mathscr{L}(E\times F,\mathbb{K})$

1. **a)** The bilinear form
$$\begin{array}{rrl} f_0: & E\times F & \longrightarrow \mathbb{K} \\ & (x,y) & \longmapsto 0_{\mathbb{K}} \end{array}$$
is the neutral element of $\mathscr{L}(E\times F,\mathbb{K})$.

   b) Let $f,g,h\in\mathscr{L}(E\times F,\mathbb{K})$. Then
$$(f+g)+h=f+(g+h),$$
thus $+$ is associative.

   c) Let $f\in\mathscr{L}(E\times F,\mathbb{K})$. We have
$$\begin{array}{rrl} -f: & E\times F & \longrightarrow \mathbb{K} \\ & (x,y) & \longmapsto -f(x,y) \end{array}$$
which is a bilinear form satisfying
$$f+(-f)=0_{\mathscr{L}(E\times F,\mathbb{K})},$$
hence every element admits an additive inverse.

   d) Let $f,g\in\mathscr{L}(E\times F,\mathbb{K})$. Then
$$f+g=g+f,$$
so $+$ is commutative.

   Therefore $(\mathscr{L}(E\times F,\mathbb{K}),+)$ is an abelian group.
2. $(\alpha\beta)\cdot g=\alpha\cdot(\beta g)$.
3. $(\alpha+\beta)\cdot f=\alpha\cdot f+\beta\cdot f$.
4. $\alpha\cdot(f+g)=\alpha\cdot f+\alpha\cdot g$.
5. $1_{\mathbb{K}}\cdot f=f$.

Hence $\mathscr{L}(E\times F,\mathbb{K})$ is a vector space over $\mathbb{K}$. ■

2.1.2 Matrix of a Bilinear Form

Let E and F be finite-dimensional vector spaces over the same field $\mathbb{K}$, $B=\{e_1,e_2,\dots,e_n\}$ a basis of E and $B'=\{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_p\}$ a basis of F.

Definition 2.1.3 Let $f \in \mathscr{L}(E \times F, \mathbb{K})$. The **matrix associated with** f with respect to B and B' is the (n,p) matrix defined by

$$a_{ij} = f(e_i, \varepsilon_j), \quad i = 1, \dots, n, \; j = 1, \dots, p,$$

in other words

$$M = \begin{pmatrix} f(e_1,\varepsilon_1) & f(e_1,\varepsilon_2) & \dots & f(e_1,\varepsilon_p) \\ f(e_2,\varepsilon_1) & f(e_2,\varepsilon_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ f(e_n,\varepsilon_1) & f(e_n,\varepsilon_2) & \dots & f(e_n,\varepsilon_p) \end{pmatrix}.$$

■ **Example 2.2 — Standard dot product on $\mathbb{K}^2$.** Let

$$E = F = \mathbb{K}^2,$$

and let

$$B = \{e_1, e_2\}, \qquad B' = \{\varepsilon_1, \varepsilon_2\}$$

be the canonical bases, where

$$e_1 = \varepsilon_1 = (1,0), \qquad e_2 = \varepsilon_2 = (0,1).$$

Define the bilinear map

$$f : E \times F \to \mathbb{K}, \qquad f(x,y) = x_1 y_1 + x_2 y_2.$$

We compute the coefficients

$$a_{ij} = f(e_i, \varepsilon_j).$$

$$\begin{aligned} a_{11} &= f(e_1,\varepsilon_1) = 1, \quad a_{12} = f(e_1,\varepsilon_2) = 0, \\ a_{21} &= f(e_2,\varepsilon_1) = 0, \quad a_{22} = f(e_2,\varepsilon_2) = 1. \end{aligned}$$

Hence the associated matrix is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

■

■ **Example 2.3 — A bilinear form on $\mathbb{K}^2 \times \mathbb{K}^3$.** Let

$$E = \mathbb{K}^2, \qquad F = \mathbb{K}^3,$$

with bases

$$B = \{e_1, e_2\}, \qquad B' = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}.$$

Define the bilinear map

$$f\big((x_1,x_2),(y_1,y_2,y_3)\big) = 2x_1y_1 + x_1y_2 + 3x_2y_3.$$

We compute the entries of the associated matrix:

For $i=1$ $(e_1=(1,0))$:

$$a_{11}=f(e_1,\varepsilon_1)=2,$$
$$a_{12}=f(e_1,\varepsilon_2)=1,$$
$$a_{13}=f(e_1,\varepsilon_3)=0.$$

For $i=2$ $(e_2=(0,1))$:

$$a_{21}=f(e_2,\varepsilon_1)=0,$$
$$a_{22}=f(e_2,\varepsilon_2)=0,$$
$$a_{23}=f(e_2,\varepsilon_3)=3.$$

Therefore, the associated matrix is

$$A=\begin{pmatrix}2 & 1 & 0\\ 0 & 0 & 3\end{pmatrix}.$$

■

2.1.3 Bilinear Form Associated with a Matrix

Definition 2.1.4 Let $M\in\mathscr{M}_{n,p}(\mathbb{K})$. The mapping

$$\begin{array}{rcl} f:\mathbb{K}^n\times\mathbb{K}^p & \longrightarrow & \mathbb{K}\\ (X,Y) & \longmapsto & X^tMY \end{array}$$

where $X=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$ et $Y=\begin{pmatrix}y_1\\y_2\\\vdots\\y_p\end{pmatrix}$ is a bilinear form called the **bilinear form associated with** M.

■ **Example 2.4** Let the matrix

$$A=\begin{pmatrix}2 & 3 & 1\\ 0 & -2 & 4\end{pmatrix}.$$

The bilinear form associated with A is

$$\begin{array}{rcl} f:\mathbb{R}^2\times\mathbb{R}^3 & \longrightarrow & \mathbb{R}\\ ((x_1,x_2),(y_1,y_2,y_3)) & \longmapsto & f\left((x_1,x_2),(y_1,y_2,y_3)\right). \end{array}$$

With

$$\begin{aligned} f\left((x_1,x_2),(y_1,y_2,y_3)\right) &= (x_1,x_2)\begin{pmatrix}2 & 3 & 1\\ 0 & -2 & 4\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\\ &= 2x_1y_1+(3x_1-2x_2)y_2+(x_1+4x_2)y_3. \end{aligned}$$

■

2.1.4 Change of Basis

Proposition 2.1.2 Let $f \in \mathscr{L}(E \times F, \mathbb{K})$. If M and N are the matrices of f with respect to bases (B, B') and (B_1, B'_1), then

$$N = {}^tPMQ,$$

where P and Q are the change-of-basis matrices.

Proof. Let $B = (e_1, \dots, e_n)$ be a basis of E and $B' = (f_1, \dots, f_m)$ a basis of F. Let M be the matrix of f with respect to (B, B'), i.e.

$$M_{ij} = f(e_i, f_j).$$

Let $B_1 = (\tilde{e}_1, \dots, \tilde{e}_n)$ and $B'_1 = (\tilde{f}_1, \dots, \tilde{f}_m)$ be new bases, and let P and Q be the change-of-basis matrices defined by

$$\tilde{e}_i = \sum_k P_{ki} e_k, \qquad \tilde{f}_j = \sum_\ell Q_{\ell j} f_\ell.$$

Let N be the matrix of f with respect to (B_1, B'_1). Then

$$N_{ij} = f(\tilde{e}_i, \tilde{f}_j).$$

Using bilinearity of f, we obtain

$$f(\tilde{e}_i, \tilde{f}_j) = f\left(\sum_k P_{ki} e_k, \sum_\ell Q_{\ell j} f_\ell\right) = \sum_{k,\ell} P_{ki} Q_{\ell j} f(e_k, f_\ell).$$

Since $f(e_k, f_\ell) = M_{k\ell}$, it follows that

$$N_{ij} = \sum_{k,\ell} P_{ki} M_{k\ell} Q_{\ell j}.$$

This is exactly the matrix product formula

$$N = {}^tPMQ.$$

■

2.2 Symmetric Bilinear Forms

Definition 2.2.1 Let $f \in \mathscr{L}(E \times E, \mathbb{K})$.

- f is called **symmetric** if

$$\forall (x, y) \in E \times E, \quad f(x, y) = f(y, x).$$

- f is called **skew-symmetric** if

$$\forall (x, y) \in E \times E, \quad f(x, y) = -f(y, x).$$

- f is called **alternating** if

$$\forall x \in E, \quad f(x, x) = 0.$$

■ **Example 2.5**

1. Let the bilinear form
$$\begin{aligned}\varphi :\quad \mathscr{C}([0,1],\mathbb{R})\times\mathscr{C}([0,1],\mathbb{R}) &\longrightarrow \mathbb{R}\\ (f,g) &\longmapsto \int_0^1 f(t)g(t)\,dt.\end{aligned}$$
Let $f,g\in\mathscr{C}([0,1],\mathbb{R})$. Then
$$\begin{aligned}\varphi(g,f) &= \int_0^1 g(t)f(t)\,dt\\ &= \int_0^1 f(t)g(t)\,dt\\ &= \varphi(f,g).\end{aligned}$$
Therefore, φ is symmetric.
2. Let the bilinear form
$$\begin{aligned}f:\quad \mathbb{R}^2\times\mathbb{R}^2 &\longrightarrow \mathbb{R}\\ ((x_1,x_2),(y_1,y_2)) &\longmapsto x_1y_2-x_2y_1.\end{aligned}$$
Let $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$. Then
$$\begin{aligned}f((y_1,y_2),(x_1,x_2)) &= y_1x_2-y_2x_1\\ &= -(x_1y_2-x_2y_1)\\ &= -f((x_1,x_2),(y_1,y_2)).\end{aligned}$$
Hence f is skew-symmetric. Moreover,
$$f((x_1,x_2),(x_1,x_2))=x_1x_2-x_2x_1=0.$$
Therefore, f is alternating.

■

Proposition 2.2.1

i) The set of symmetric bilinear forms, denoted $\mathscr{S}(E\times E,\mathbb{K})$, is a vector subspace of $\mathscr{L}(E\times E,\mathbb{K})$.
ii) The set of skew-symmetric bilinear forms, denoted $\mathscr{A}(E\times E,\mathbb{K})$, is a vector subspace of $\mathscr{L}(E\times E,\mathbb{K})$.

Proof.

i) **a)** We have $f_0(x,y)=f_0(y,x)=O_{\mathbb{K}}$, so
$$f_0\in\mathscr{S}(E\times E,\mathbb{K}). \tag{2.1}$$

b) Let $f,g\in\mathscr{S}(E\times E,\mathbb{K})$ and $\alpha,\beta\in\mathbb{K}$. Then
$$\begin{aligned}(\alpha f+\beta g)(x,y) &= \alpha f(x,y)+\beta g(x,y)\\ &= \alpha f(y,x)+\beta g(y,x)\\ &= (\alpha f+\beta g)(y,x).\end{aligned}$$
Hence
$$\alpha f+\beta g\in\mathscr{S}(E\times E,\mathbb{K}). \tag{2.2}$$

From (2.1) and (2.2), $\mathscr{S}(E\times E,\mathbb{K})$ is a vector subspace of $\mathscr{L}(E\times E,\mathbb{K})$.

ii) **a)** We have $f_0(x,y)=-f_0(y,x)=O_{\mathbb{K}}$, so
$$f_0\in\mathscr{A}(E\times E,\mathbb{K}). \tag{2.3}$$

b) Let $f,g \in \mathscr{A}(E\times E,\mathbb{K})$ and $\alpha,\beta \in \mathbb{K}$. Then

$$\begin{aligned}(\alpha f+\beta g)(x,y) &= \alpha f(x,y)+\beta g(x,y)\\ &= -\alpha f(y,x)-\beta g(y,x)\\ &= -(\alpha f+\beta g)(y,x).\end{aligned}$$

Hence

$$\alpha f+\beta g \in \mathscr{A}(E\times E,\mathbb{K}). \tag{2.4}$$

From (2.3) and (2.4), $\mathscr{A}(E\times E,\mathbb{K})$ is a vector subspace of $\mathscr{L}(E\times E,\mathbb{K})$.

■

Proposition 2.2.2 We have

$$\mathscr{L}(E\times E,\mathbb{K}) = \mathscr{S}(E\times E,\mathbb{K}) \oplus \mathscr{A}(E\times E,\mathbb{K}).$$

Proof.

i) Let $f \in \mathscr{S}(E\times E,\mathbb{K})\cap\mathscr{A}(E\times E,\mathbb{K})$. Then

$$f(x,y)=f(y,x), \quad \forall x,y\in E$$

and

$$f(x,y)=-f(y,x), \quad \forall x,y\in E.$$

Hence, $f(x,y)=0$ for all $x,y\in E$, and consequently

$$\mathscr{S}(E\times E,\mathbb{K})\cap\mathscr{A}(E\times E,\mathbb{K}) = \{0_E\}. \tag{2.5}$$

ii) Clearly,

$$\mathscr{S}(E\times E,\mathbb{K})+\mathscr{A}(E\times E,\mathbb{K}) \subset \mathscr{L}(E\times E,\mathbb{K}).$$

On the other hand, let $f \in \mathscr{L}(E\times E,\mathbb{K})$. Define

$$f(x,y) = \underbrace{\frac{1}{2}\big(f(x,y)+f(y,x)\big)}_{f_1(x,y)} + \underbrace{\frac{1}{2}\big(f(x,y)-f(y,x)\big)}_{f_2(x,y)}.$$

Then $f_1 \in \mathscr{S}(E\times E,\mathbb{K})$ and $f_2 \in \mathscr{A}(E\times E,\mathbb{K})$. Hence

$$\mathscr{L}(E\times E,\mathbb{K}) \subset \mathscr{S}(E\times E,\mathbb{K})+\mathscr{A}(E\times E,\mathbb{K}),$$

and therefore

$$\mathscr{L}(E\times E,\mathbb{K}) = \mathscr{S}(E\times E,\mathbb{K})+\mathscr{A}(E\times E,\mathbb{K}). \tag{2.6}$$

From (2.5) and (2.6), we conclude

$$\mathscr{L}(E\times E,\mathbb{K}) = \mathscr{S}(E\times E,\mathbb{K}) \oplus \mathscr{A}(E\times E,\mathbb{K}).$$

■

2.2.1 Kernel of a Symmetric Bilinear Form

Definition 2.2.2 Let $f \in \mathscr{L}(E \times E, \mathbb{K})$. The **kernel** of f, denoted $Ann(f)$, is

$$Ann(f) = \{x \in E : \ f(x,y) = 0, \ \forall y \in E\}.$$

2.2.2 Nondegenerate Symmetric Bilinear Forms

Definition 2.2.3 f is said to be **nondegenerate** if $Ann(f) = \{0_E\}$.

Example 2.6 Consider the symmetric bilinear form

$$\begin{array}{llll} g: & \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & ((x_1,x_2),(y_1,y_2)) & \longmapsto & (x_1+x_2)(y_1+y_2). \end{array}$$

The kernel (annihilator) of g is

$$\text{Ann}(g) = \{(x_1,x_2) \in \mathbb{R}^2 \mid \forall (y_1,y_2) \in \mathbb{R}^2, \ g((x_1,x_2),(y_1,y_2)) = 0\}.$$

We have

$$g((x_1,x_2),(y_1,y_2)) = 0 \quad \forall (y_1,y_2) \in \mathbb{R}^2.$$

- For $(y_1,y_2) \neq (0,0)$, this implies $x_1 = -x_2$.

Hence,

$$\text{Ann}(g) = \{(x_1,-x_1) \mid x_1 \in \mathbb{R}\} = \langle (1,-1) \rangle.$$

Since $\text{Ann}(g) \neq \{(0,0)\}$, the bilinear form g is degenerate. ■

Example 2.7 Consider the bilinear form

$$f : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad f((x_1,x_2,x_3),(y_1,y_2,y_3)) = x_1y_1 + 2x_2y_2 + 3x_3y_3.$$

The kernel (annihilator) of f is

$$\text{Ann}(f) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid f((x_1,x_2,x_3),(y_1,y_2,y_3)) = 0, \ \forall (y_1,y_2,y_3) \in \mathbb{R}^3\}.$$

Since

$$f((x_1,x_2,x_3),(1,0,0)) = x_1, \quad f((x_1,x_2,x_3),(0,1,0)) = 2x_2, \quad f((x_1,x_2,x_3),(0,0,1)) = 3x_3,$$

we must have $x_1 = x_2 = x_3 = 0$ for $f((x_1,x_2,x_3),y) = 0$ to hold for all y.

Hence,

$$\text{Ann}(f) = \{(0,0,0)\}.$$

This shows that the kernel of a non-degenerate bilinear form is trivial. ■

Proposition 2.2.3 Let $f \in \mathscr{L}(E \times E, \mathbb{K})$. Consider the map

$$\Phi : E \longrightarrow E^* \quad \text{defined by} \quad \forall y \in E, \ \Phi(y) = \varphi_y, \ \text{where } \forall x \in E, \ \varphi_y(x) = f(x,y).$$

Then Φ is injective if and only if f is non-degenerate.

Proof. The map Φ is linear, and we have

$$\begin{aligned} \text{Ker}(\Phi) &= \{x \in E \mid \Phi(x) = 0\} \\ &= \{x \in E \mid \varphi_y(x) = 0, \ \forall y \in E\} \\ &= \{x \in E \mid f(x,y) = 0, \ \forall y \in E\} = \text{Ann}(f). \end{aligned}$$

Hence, Φ is injective if and only if $\text{Ker}(\Phi) = \{0_E\} = \text{Ann}(f)$. Consequently, f is non-degenerate. ■

Corollary 2.2.4 Let $f \in \mathscr{L}(E \times E, \mathbb{K})$, where E is finite-dimensional, B is a basis of E, and A is the matrix of f with respect to the basis B. Then

$$f \text{ is non-degenerate } \Leftrightarrow \det(A) \neq 0.$$

Proof. Consider the map $\Phi : E \to E^*$ defined by

$$\forall y \in E, \quad \Phi(y) = \varphi_y, \quad \text{where} \quad \forall x \in E, \ \varphi_y(x) = f(x,y).$$

Let $B = \{e_1, e_2, \dots, e_n\}$ be a basis of E, and $B^* = \{e_1^*, e_2^*, \dots, e_n^*\}$ its dual basis.
Let $M = (m_{ij})_{1 \leq i,j \leq n}$ be the matrix of Φ with respect to the bases B and B^*. Then

$$\forall j = 1, \dots, n, \quad \Phi(e_j) = \sum_{k=1}^{n} m_{kj} e_k^*.$$

Hence, for each $i, j \in \{1, \dots, n\}$,

$$f(e_i, e_j) = \Phi(e_j)(e_i) = \sum_{k=1}^{n} m_{kj} e_k^*(e_i) = m_{ij}, \quad \text{since } e_k^*(e_i) = \delta_{ik}.$$

It follows that $M = A$. Therefore,

$$\begin{aligned} f \text{ is non-degenerate } &\Leftrightarrow \Phi \text{ is injective (Proposition 2.2.3)} \\ &\Leftrightarrow \Phi \text{ is bijective (since } \dim E = \dim E^*) \\ &\Leftrightarrow \det(M) \neq 0 \\ &\Leftrightarrow \det(A) \neq 0 \quad (\text{since } M = A). \end{aligned}$$

■

2.3 Orthogonality

2.3.1 Orthogonal Bases

Definition 2.3.1 A family $\mathscr{F}$ of E is said to be *orthogonal* for f (or simply orthogonal if there is no ambiguity about f) if for all $x, y \in \mathscr{F}$, with $x \neq y$, we have $x \perp_f y$. In other words, if

$$f(x,y) = 0_K \qquad (\forall x, y \in \mathscr{F}, \ x \neq y).$$

— When a basis $\mathscr{B}$ of E is an orthogonal family for f, we say that $\mathscr{B}$ is an *orthogonal basis* of E for f (or simply an orthogonal basis of E if there is no ambiguity about f).

— The expression "orthogonal for f" is sometimes replaced by one of the following expressions: "orthogonal for q_f", "f-orthogonal", or "q_f-orthogonal".

When E is finite-dimensional, determining an orthogonal basis of E for f is very useful for simplifying and classifying the symmetric bilinear form f; moreover, such a basis always exists!

Theorem 2.3.1 Suppose that E is finite-dimensional. Then there exists at least one orthogonal basis of E for f.

2.4 Orthogonal matrices

Definition 2.4.1 A square matrix $A \in \mathbb{R}^{n\times n}$ is called **orthogonal** if

$$A^T A = AA^T = I_n,$$

where A^T denotes the transpose of A and I_n is the $n \times n$ identity matrix.

Equivalently, a matrix is orthogonal if its columns (or rows) form an orthonormal set of vectors with respect to the standard Euclidean inner product.

2.4.1 Properties of Orthogonal Matrices

Orthogonal matrices possess several important properties:

Proposition 2.4.1 Let $A \in \mathbb{R}^{n\times n}$ be orthogonal. Then:

1. $A^{-1} = A^T$ (the inverse equals the transpose).
2. $\det(A) = \pm 1$.
3. Orthogonal matrices preserve the Euclidean norm: for any $x \in \mathbb{R}^n$,

$$\|Ax\| = \|x\|.$$

4. The columns (and rows) of A form an orthonormal basis of $\mathbb{R}^n$:

$$\langle a_i, a_j \rangle = \delta_{ij}.$$

Theorem 2.4.2 — Preservation of Inner Product. If A is orthogonal and $x, y \in \mathbb{R}^n$, then

$$\langle Ax, Ay \rangle = \langle x, y \rangle.$$

Proof.

$$\langle Ax, Ay \rangle = (Ax)^T (Ay) = x^T A^T Ay = x^T I_n y = x^T y = \langle x, y \rangle.$$

■

2.5 Examples

■ **Example 2.8** The identity matrix I_n is orthogonal because $I_n^T I_n = I_n$. ■

■ **Example 2.9** The rotation matrix in $\mathbb{R}^2$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is orthogonal, as $R_\theta^T R_\theta = I_2$, and $\det(R_\theta) = 1$. ■

■ **Example 2.10** The reflection matrix in $\mathbb{R}^2$ over the x-axis

$$F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is orthogonal, since $F^T F = I_2$ and $\det(F) = -1$. ■