1. Linear Forms, Duality

Introduction

This chapter introduces the notion of linear forms and the concept of duality in vector spaces. These notions play an important role in linear algebra and appear naturally in many mathematical problems.

A linear form is a linear mapping from a vector space into the base field $\mathbb{R}$ or $\mathbb{C}$. Linear forms allow us to describe hyperplanes and provide a useful tool for studying vector spaces from a new point of view.

We then define the dual space of a vector space as the set of all linear forms on that space. In the finite-dimensional case, the dual space has the same dimension as the original space, and each basis admits a corresponding dual basis.

Finally, we introduce the bidual space and show that every finite-dimensional vector space can be identified in a natural way with its bidual. This chapter provides the basic tools needed for the study of more advanced topics in linear algebra and related fields.

Chapter Objectives

- Introduce the concept of **linear forms** and their basic properties.
- Explain the relation between **linear forms and hyperplanes**.
- Present the notion of the **dual vector space**.
- Introduce the **dual basis and bidual space**.

1.1 Linear Forms

In the following, E denotes a vector space over $K = \mathbb{R}$ or $\mathbb{C}$, of finite or infinite dimension.

Definition 1.1.1 A linear form on E is a linear mapping from E into K.

Example 1.1 1. If $E = \mathscr{C}([0,1], K)$ is the vector space of continuous functions on $[0,1]$, the mapping

$$\begin{array}{rccl} \Psi: & E & \longrightarrow & K \\ & f & \longmapsto & \int_0^1 f(t)\,dt \end{array}$$

is a linear form on E.

2. Let $\mathscr{M}_n(K)$ be the vector space of square matrices of order n with coefficients in K. The mapping

$$\begin{array}{rccl} Tr: & \mathscr{M}_n(K) & \longrightarrow & K \\ & M & \longmapsto & Tr(M) \end{array}$$

is a linear form on $\mathscr{M}_n(K)$.

3. If E has dimension n and $\mathbf{B} = (e_j)_{1\leq j\leq n}$ is a basis of E, then the coordinate projections relative to $\mathbf{B}$

$$\begin{array}{rccl} p_j: & E & \longrightarrow & K \\ & x = \sum_{i=1}^{n} x_i e_i & \longmapsto & x_j \end{array}$$

are linear forms on E.

■

1.2 Hyperplanes

Hyperplanes are vector subspaces defined as kernels of nonzero linear forms. In finite-dimensional spaces, a hyperplane is a subspace whose dimension is one less than that of the ambient space.

Proposition 1.2.1 Let f be a nonzero linear form on a vector space E of dimension n. Then

$$\dim \ker f = n - 1.$$

Proof. By the Rank Theorem, we have

$$\dim \ker f + \dim \operatorname{Im} f = \dim E.$$

Since f is nonzero, $\operatorname{Im} f \neq \{0_K\}$ and $\operatorname{Im} f$ is a subspace of K. Thus

$$0 < \dim \operatorname{Im} f \leq \dim K = 1,$$

which implies $\dim \operatorname{Im} f = 1$, and therefore

$$\dim \ker f = n - 1.$$

■

Proposition (1.2.1) leads to a generalization of the notion of hyperplane to vector spaces of finite or infinite dimension.

Definition 1.2.1 A hyperplane of E is a vector subspace $H \subset E$ such that there exists a nonzero linear form $f : E \to K$ satisfying $H = \ker f$.

If $\dim E = n < \infty$, Proposition (1.2.1) recovers the classical definition of a hyperplane: a subspace of dimension $n-1$.

■ **Example 1.2** 1. The set

$$H = \{(x,y,z) \in \mathbb{R}^3 : x + 2y - z = 0\}$$

is a hyperplane of $\mathbb{R}^3$, since the mapping

$$f(x,y,z) = x + 2y - z$$

is a nonzero linear form and $\ker f = H$.

2. The set

$$H = \{f \in C([0,1],\mathbb{R}) : f(0) = 0\}$$

is a hyperplane of $C([0,1],\mathbb{R})$, since the mapping

$$g(f) = f(0)$$

is a nonzero linear form and $\ker g = H$.

■

Corollary 1.2.2 Let H be a hyperplane of E. Then for any one-dimensional vector subspace D of E not contained in H, we have

$$E = H \oplus D.$$

R By definition, a one-dimensional vector subspace is called a vector line.

1.3 Duality

Duality studies a vector space through the linear forms defined on it. The dual space consists of all linear forms, and in finite dimension it has the same dimension as the original space, with a natural associated dual basis.

1.3.1 Dual Vector Space

Definition 1.3.1 The dual vector space of E, denoted E^*, is the vector space of all linear forms on E:

$$E^* = \mathscr{L}(E,K).$$

Corollary 1.3.1 If E is finite-dimensional, then E^* is also finite-dimensional and

$$\dim E = \dim E^*.$$

Proof. If E is finite-dimensional, then $\mathscr{L}(E,K)$ is also finite-dimensional and

$$\dim E^* = \dim \mathscr{L}(E,K) = \dim E \times \dim K = \dim E.$$

■

1.3.2 Dual Basis

Assume E is finite-dimensional and let $\mathbf{B} = \{e_1, e_2, \dots, e_n\}$ be a basis of E. Define the family $\mathbf{B}^* = \{e_1^*, e_2^*, \dots, e_n^*\}$ in E^* by

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.1}$$

Theorem 1.3.2 Let E be a vector space of dimension n. Then its dual E^* has dimension n, and for any basis $\mathbf{B} = \{e_1, e_2, \dots, e_n\}$ of E, the family $\mathbf{B}^* = \{e_1^*, e_2^*, \dots, e_n^*\}$ is a basis of E^*, called the dual basis of $\mathbf{B}$.

Proof. Since $\dim E^* = \text{Card}(\mathbf{B}^*) = n$, it suffices to show that the family $\{e_1^*, e_2^*, \dots, e_n^*\}$ is linearly independent. To this end, let $\alpha_1, \alpha_2, \dots, \alpha_n \in K$ such that

$$\alpha_1 e_1^* + \alpha_2 e_2^* + \dots + \alpha_n e_n^* = 0.$$

For every $j \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} (\alpha_1 e_1^* + \alpha_2 e_2^* + \dots + \alpha_n e_n^*)(e_j) &= 0, \\ \alpha_1 e_1^*(e_j) + \alpha_2 e_2^*(e_j) + \dots + \alpha_j e_j^*(e_j) + \dots + \alpha_n e_n^*(e_j) &= 0, \\ \alpha_j &= 0. \end{aligned}$$

Hence $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$, and therefore $\{e_1^*, e_2^*, \dots, e_n^*\}$ forms a basis of E^*. ■

■ **Example 1.3** In $\mathbb{R}^2$, consider the basis

$$B = \{v_1 = (1,1), v_2 = (0,1)\}.$$

Determine the dual basis of B.

Let $B^* = \{f_1, f_2\}$ be the dual basis, where

$$f_i(x,y) = a_i x + b_i y.$$

Solving the conditions $f_i(v_j) = \delta_{ij}$ yields

$$f_1(x,y) = x, \qquad f_2(x,y) = y - x.$$

■

1.3.3 Bidual of a Vector Space

Definition 1.3.2 Let E be a vector space over K. The bidual of E, denoted E^{**}, is the dual of E^*:

$$E^{**} = (E^*)^* = \mathscr{L}(E^*, K).$$

Proposition 1.3.3 If E is finite-dimensional, then the mapping

$$\begin{aligned} \widetilde{\ }: \quad E &\longrightarrow E^{**} \\ x &\longmapsto \widetilde{x} \end{aligned}$$

defined by $\widetilde{x}(f) = f(x)$ for all $f \in E^*$ is a linear bijection. This mapping is called the **canonical isomorphism** between E and E^{**}.

Proof. 1. For every fixed $x \in E$, the mapping

$$\begin{aligned} \widetilde{x}: \quad E^* &\longrightarrow K \\ f &\longmapsto f(x) \end{aligned}$$

is linear; hence $\widetilde{x} \in E^{**}$.

2. Let $x, y \in E$, $\alpha \in K$, and $f \in E^*$. We have

$$\widetilde{(x+\alpha y)}(f) = f(x+\alpha y) = f(x) + \alpha f(y) = \widetilde{x}(f) + \alpha \widetilde{y}(f).$$

Therefore, the mapping $\sim$ is linear.

3. Since $\dim E = \dim E^* = \dim E^{**}$, it suffices to show that $\sim$ is injective, that is, $\ker(\sim) = \{0_E\}$.

$$\begin{aligned} \ker(\sim) &= \{x \in E : \widetilde{x} = 0_{E^{**}}\} \\ &= \{x \in E : f(x) = 0 \text{ for all } f \in E^*\} \\ &= \{0_E\}. \end{aligned}$$

Indeed, if $x \neq 0$, one can choose a basis $\{e_1, e_2, \ldots, e_n\}$ of E such that $e_1 = x$. Then the dual vector $e_1^* \in E^*$ satisfies $e_1^*(x) = 1 \neq 0$, which is a contradiction. Hence $\widetilde{x} = 0_{E^{**}}$ if and only if $f(x) = 0$ for all $f \in E^*$.

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