Analysis II: Solutions of Tutorial Exercise Sheet 1

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2025-2026

Exercise 01: Classification of Differential Equations

(1) Order 2, dep. var. y, indep. var. t, ordinary.

(2) Order 2, dep. var. u, indep. var. x and t, partial.

(3) Order 1, dep. var. x, indep. var. y, ordinary.

(4) Order 3, dep. var. r, indep. var. θ, ordinary.

(5) Order 1, dep. var. y (or x), indep. var. x (or y), ordinary.

Exercise 02: Solutions of Differential Equations

(1) **Yes, General Solution.**
Substitute derivatives: $-9y + 9y = 0$.
Order is 2, and there are 2 arbitrary constants (C_1, C_2).

(2) **Yes, Not a General Solution.**
Substitute: $x(10x) - 2(5x^2) = 10x^2 - 10x^2 = 0$.
It is a solution, but it has 0 arbitrary constants while the DE is of order 1. (It is a particular solution).

(3) **Yes, Not a General Solution.**
Substitute: $Ce^x - Ce^x = 0$.
It is a solution, but the DE is of order 2 while the solution has only 1 constant.

(4) **Yes, General Solution.**
$y' = \cos(x + C)$. Check: $\cos^2(x + C) + \sin^2(x + C) = 1$.
Order is 1, and there is 1 arbitrary constant (C).

Exercise 03: Initial Value Problems (IVP)

(1) Substitute $x = 0, y = 5$: $5 = Ce^0 \Rightarrow C = 5$.
Particular Solution: $y = 5e^{-2x}$

(2) (i) $y(0) = 4 \Rightarrow C_1 + C_2 = 4$.
(ii) $y' = C_1e^x - C_2e^{-x}$. So, $y'(0) = 0 \Rightarrow C_1 - C_2 = 0 \Rightarrow C_1 = C_2$.
Solving (i) & (ii): $2C_1 = 4 \Rightarrow C_1 = 2, C_2 = 2$.
Particular Solution: $y = 2e^x + 2e^{-x}$ (or $4\cosh x$)

(3) (i) $x(0) = A \Rightarrow C_1(1) + 0 = A \Rightarrow C_1 = A$.
(ii) $x'(t) = -C_1\omega\sin(\omega t) + C_2\omega\cos(\omega t)$.
$x'(0) = 0 \Rightarrow 0 + C_2\omega(1) = 0 \Rightarrow C_2 = 0$.
Particular Solution: $x(t) = A\cos(\omega t)$

(4) It is a solution, but **not** the general solution. The DE is of order 1, so the general solution must contain 1 arbitrary constant (e.g., $y = x^3 + C$).

Exercise 04: Verification of Initial Value Problems

(1) For DE 1:

- *Derivatives:*
  $y = x\sin x$
  $y' = \sin x + x\cos x$
  $y'' = \cos x + (\cos x - x\sin x) = 2\cos x - x\sin x$
- *Verification of DE:*
  Substitute into $y'' + y$:
  $(2\cos x - x\sin x) + (x\sin x) = 2\cos x$. (Matches RHS)

– *Verification of Conditions:*
$y(0) = 0 \cdot \sin(0) = 0.$ (OK)
$y'(0) = \sin(0) + 0 \cdot \cos(0) = 0.$ (OK)

(2) For DE 2:

– *Derivatives:*
$y = 3e^{2t} - 2 \implies y' = 6e^{2t}.$

– *Verification of DE:*
Substitute into $y' - 2y$:
$6e^{2t} - 2(3e^{2t} - 2) = 6e^{2t} - 6e^{2t} + 4 = 4.$ (Matches RHS)

– *Verification of Condition:*
$y(0) = 3e^0 - 2 = 3(1) - 2 = 1.$ OK

Exercise 05: Separation of Variables

1. $\frac{dy}{dx} - 2x = 0$
Separating the variables, we have $dy = 2x\,dx$. Integrating, $\int dy = \int 2x\,dx$ i.e. $y = x^2 + c$.

2. $\frac{dy}{dx} + 3y = 0$
Separating the variables, we have $\frac{dy}{y} = -3\,dx$. Integrating, $\int \frac{dy}{y} = \int -3\,dx$ i.e. $\ln y = -3x + c_1$. This can also be written as $y = ce^{-3x}$.

3. $y'(x^2 + 1) - 2xy = 0$
The equation can be written as $(x^2 + 1)\frac{dy}{dx} = 2xy$ or $\frac{dy}{y} = \frac{2x}{x^2+1}dx$.
Integrating, $\int \frac{dy}{y} = \int \frac{2x}{x^2+1}dx$
i.e. $\ln y = \ln(x^2 + 1) + \ln c$.
Thus the solution is $y = c(x^2 + 1)$.

4. $\sqrt{x^2 + 1}\,dy = xy\,dx$
Separating the variables, we have $\frac{dy}{y} = \frac{x}{\sqrt{x^2+1}}dx$. Integrating, $\int \frac{dy}{y} = \int \frac{x}{\sqrt{x^2+1}}dx$ i.e. $\ln y = \sqrt{x^2 + 1} + c$.

5. $y\,dx - x\,dy = 0$
Separating the variables, $\frac{dx}{x} = \frac{dy}{y}$. Integrating, $\ln x = \ln y + \ln c$. i.e. $x = cy$.

6. $(1 + u)v\,du + (1 - v)u\,dv = 0$
Separating the variables, $\frac{1+u}{u}du + \frac{1-v}{v}dv = 0$ or $(\frac{1}{u} + 1)du + (\frac{1}{v} - 1)dv = 0$. Integrating, $\ln u + u + \ln v - v = c$. i.e. $\ln(uv) + u - v = c$.

7. $(1 + y)\,dx - (1 - x)\,dy = 0$
Separating the variables, $\frac{dx}{1-x} - \frac{dy}{1+y} = 0$. Integrating, $-\ln(1-x) - \ln(1+y) = c_1$. i.e. $(1-x)(1+y) = c$.

8. $(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0$
Factorizing: $x^2(1 - y)\frac{dy}{dx} + y^2(1 + x) = 0$. Separating the variables, $(\frac{1-y}{y^2})dy + (\frac{1+x}{x^2})dx = 0$. Integrating, $-\frac{1}{y} - \ln y - \frac{1}{x} + \ln x = c$. i.e. $\ln(x/y) - \frac{1}{x} - \frac{1}{y} = c$.

9. $(y - a)\,dx + x^2\,dy = 0$
Separating the variables, $\frac{dx}{x^2} + \frac{dy}{y-a} = 0$. Integrating, $-\frac{1}{x} + \ln(y - a) = c$.

10. $z\,dt - (t^2 - a^2)\,dz = 0$
Separating the variables, $\frac{dt}{t^2-a^2} = \frac{dz}{z}$. Integrating, $\frac{1}{2a}\ln\left(\frac{t-a}{t+a}\right) = \ln z + c_1$.

11. $\frac{dx}{dy} = \frac{1+x^2}{1+y^2}$
Separating the variables, $\frac{dx}{1+x^2} = \frac{dy}{1+y^2}$. Integrating, $\arctan x = \arctan y + c$.

12. $(1+s^2)\,dt - \sqrt{t}\,ds = 0$
Separating the variables, $\frac{dt}{\sqrt{t}} = \frac{ds}{1+s^2}$. Integrating, $2\sqrt{t} = \arctan s + c$.

13. $(1+x^2)\,dy - \sqrt{1-y^2}\,dx = 0$
Separating the variables, $\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{1+x^2}$. Integrating, $\arcsin y = \arctan x + c$.

14. $\sqrt{1-x^2}\,dy - \sqrt{1-y^2}\,dx = 0$. Separating the variables, $\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$. Integrating, $\arcsin y = \arcsin x + c$.

15. $(t - y^2 t)\,dt + (y - t^2 y)\,dy = 0$
Separating variables: $\frac{t}{1-t^2}dt + \frac{y}{1-y^2}dy = 0$. Integrating, $-\frac{1}{2}\ln(1-t^2) - \frac{1}{2}\ln(1-y^2) = c_1$. i.e. $(1-t^2)(1-y^2) = c$.

Exercise 06: Separation of Variables with conditions

1. $(1+e^{-x})y' = e^{-x}, \quad y(0) = 0$.
Separating the variables, we have $dy = \frac{e^{-x}}{1+e^{-x}}dx$.
Integrating, $\int dy = \int \frac{e^{-x}}{1+e^{-x}}dx$
i.e. $y = -\ln(1+e^{-x}) + c$.
For the particular solution where $y(0) = 0$, i.e. $y = 0$ when $x = 0$, put $x = 0, y = 0$:
$0 = -\ln(1+1) + c \implies c = \ln 2$.
Thus the required solution is $y = -\ln(1+e^{-x}) + \ln 2$ or $y = \ln\left(\frac{2}{1+e^{-x}}\right)$.

2. $(4x + xy^2)dx + (y + x^2 y)dy = 0, \quad y(1) = 2$.
The equation can be written as $x(4+y^2)dx + y(1+x^2)dy = 0$ or $\frac{x}{1+x^2}dx + \frac{y}{4+y^2}dy = 0$.
Integrating, $\frac{1}{2}\ln(1+x^2) + \frac{1}{2}\ln(4+y^2) = c_1$.
i.e. $\ln(1+x^2)(4+y^2) = c$.
Thus the general solution is $(1+x^2)(4+y^2) = C$.
For the particular solution where $y(1) = 2$, put $x = 1, y = 2$:
$(1+1)(4+4) = C \implies C = 16$.
Thus the required solution is $(1+x^2)(4+y^2) = 16$.

3. $2xy\,dx + (x^2+1)\,dy = 0, \quad y(0) = 2$
Separating the variables, $\frac{2x}{x^2+1}dx + \frac{dy}{y} = 0$.
Integrating, $\ln(x^2+1) + \ln y = \ln c$.
i.e. $y(x^2+1) = c$.
For the particular solution where $y(0) = 2$, put $x = 0, y = 2$:
$2(0+1) = c \implies c = 2$.
Thus the required solution is $y(x^2+1) = 2$.

4. $\frac{dy}{dx} + 3y = 8, \quad y(0) = 2$ and $y(0) = 4$.
We have $dy/dx = 8 - 3y$ so that on separating the variables, $\frac{dy}{8-3y} = dx$.
Integrating, $-\frac{1}{3}\ln|8-3y| = x + c$.
(i) For $y(0) = 2$:
$-\frac{1}{3}\ln|8-6| = 0 + c \implies c = -\frac{1}{3}\ln 2$.
The solution is $-\frac{1}{3}\ln(8-3y) = x - \frac{1}{3}\ln 2$, which gives $y = \frac{1}{3}(8 - 2e^{-3x})$.
(ii) For $y(0) = 4$:
$-\frac{1}{3}\ln|8-12| = 0 + c \implies c = -\frac{1}{3}\ln 4$.
The solution is $-\frac{1}{3}\ln(3y-8) = x - \frac{1}{3}\ln 4$, which gives $y = \frac{1}{3}(8 + 4e^{-3x})$.

Exercise 07

1. $(2x+y)dx - xdy = 0$
To solve the homogeneous differential equation $y' = \frac{2x+y}{x} = 2 + \frac{y}{x}$, we use the auxiliary function

$$t(x) = \frac{y}{x}$$

which implies $y = tx$ and $y' = t'x + t$.
Substituting y and y' into the differential equation, we obtain another differential equation with separated variables:

$$t'x + t = 2 + t \implies t' = \frac{2}{x}$$

which can be rewritten as

$$\frac{dt}{2} = \frac{dx}{x}$$

Integrating, we get $t = 2\ln|x| + c$.
Finally, we substitute t back as $\frac{y}{x}$:

$$\frac{y}{x} = 2\ln|x| + c \implies y = x(2\ln|x| + c).$$

2. $(2x^3 + y^3)\,dx - 3xy^2\,dy = 0$
The homogeneous equation is $y' = \frac{2x^3+y^3}{3xy^2} = \frac{2}{3(y/x)^2} + \frac{y/x}{3}$. Let $f(t) = \frac{2}{3t^2} + \frac{t}{3}$.
Using the auxiliary function $t = y/x$, we have $y' = t'x + t$.
Substituting, we obtain:

$$t'x + t = \frac{2}{3t^2} + \frac{t}{3} \implies t'x = \frac{2}{3t^2} - \frac{2t}{3} = \frac{2(1-t^3)}{3t^2}$$

which can be rewritten as

$$\frac{3t^2}{2(1-t^3)}dt = \frac{1}{x}dx$$

Integrating, $-\frac{1}{2}\ln|1-t^3| = \ln|x| + c_1$.
Finally, substituting $t = y/x$ and simplifying:

$$x^3(1 - y^3/x^3) = c \implies x^3 - y^3 = cx.$$

3. $\frac{dy}{dx} = \frac{5x+4y}{2x-y}$
To solve homogeneous DE: $y' = \frac{5+4(y/x)}{2-(y/x)}$, let $f(t) = \frac{5+4t}{2-t}$.
Using $t = y/x$, we have $y' = t'x + t$.
Substituting y and y':

$$t'x + t = \frac{5+4t}{2-t} \implies t'x = \frac{5+4t-t(2-t)}{2-t} = \frac{t^2+2t+5}{2-t}$$

which can be rewritten as

$$\frac{2-t}{t^2+2t+5}dt = \frac{1}{x}dx$$

For the left-hand side, complete the square: $t^2 + 2t + 5 = (t+1)^2 + 4$. Substitute $u = t + 1$, then $2 - t = 3 - u$:

$$\int \frac{3-u}{u^2+4}\,du = 3\int \frac{du}{u^2+4} - \int \frac{u}{u^2+4}\,du = \frac{3}{2}\arctan\frac{u}{2} - \frac{1}{2}\ln(u^2+4) + C.$$

Back-substituting $u = t + 1$ gives

$$\int \frac{2-t}{t^2+2t+5}\,dt = \frac{3}{2}\arctan\frac{t+1}{2} - \frac{1}{2}\ln(t^2+2t+5) + C.$$

The right-hand side integrates to $\ln|x| + C$. Hence the implicit solution is

$$\frac{3}{2}\arctan\frac{t+1}{2} - \frac{1}{2}\ln(t^2+2t+5) = \ln|x| + C.$$

Finally, we substitute t back as $\frac{y}{x}$ after integration.

4. $\frac{dy}{dx} + 3x^2y = 6x^2$

   (a) Compute the integrating factor: $\mu(x) = e^{\int 3x^2\,dx} = e^{x^3}$.

(b) Multiply the equation by $\mu(x)$: $e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$.

(c) This simplifies to: $\frac{d}{dx}(e^{x^3}y) = 6x^2e^{x^3}$.

(d) Integrate: $e^{x^3}y = \int 6x^2e^{x^3}\,dx + C$.
Let $u = x^3$, then $du = 3x^2dx$. The integral becomes $\int 2e^u du = 2e^{x^3} + C$.

(e) Solve for y: $y = \frac{1}{e^{x^3}}(2e^{x^3} + C) = 2 + Ce^{-x^3}$.

5. $y' + 2xy = 1$

   (a) Compute the integrating factor: $\mu(x) = e^{\int 2x dx} = e^{x^2}$.

   (b) Multiply the equation by $\mu(x)$: $e^{x^2}y' + 2xe^{x^2}y = e^{x^2}$.

   (c) This simplifies to: $\frac{d}{dx}(e^{x^2}y) = e^{x^2}$.

   (d) Integrate: $e^{x^2}y = \int e^{x^2}\,dx + C$.

   (e) Solve for y: $y = \frac{1}{e^{x^2}}(\int e^{x^2}\,dx + C) = e^{-x^2}\int e^{x^2}\,dx + Ce^{-x^2}$.

6. $x\frac{dy}{dx} - 2y = x^3\cos 4x$
   First, divide by x to get the standard form $\frac{dy}{dx} + P(x)y = Q(x)$:

$$\frac{dy}{dx} - \frac{2}{x}y = x^2\cos 4x$$

   (a) Compute the integrating factor: $\mu(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = x^{-2}$.

   (b) Multiply the equation by $\mu(x)$: $x^{-2}\frac{dy}{dx} - 2x^{-3}y = \cos 4x$.

   (c) This simplifies to: $\frac{d}{dx}(x^{-2}y) = \cos 4x$.

   (d) Integrate: $x^{-2}y = \int \cos 4x\,dx + C = \frac{1}{4}\sin 4x + C$.

   (e) Solve for y: $y = x^2(\frac{1}{4}\sin 4x + C) = \frac{1}{4}x^2\sin 4x + Cx^2$.

Exercise 08 : Solutions of Linear First-Order Differential Equations-Variation of Parameters (First Method)

1. $\frac{dx}{dt} - x = t + 2t^2$

   - Homogeneous solution: $\dfrac{dx}{dt} - x = 0 \;\Rightarrow\; x_h = Ce^t$.
   - Particular solution: assume $x_p = u(t)e^t$. Then $x_p' = u'e^t + ue^t$. Substituting: $u'e^t = t + 2t^2 \;\Rightarrow\; u'(t) = (t + 2t^2)e^{-t}$.
   - Integrate $u'(t)$ using the tabular method (differentiate the polynomial, integrate e^{-t}):

Differentiate $(t + 2t^2)$	Integrate e^{-t}	Sign
$2t^2 + t$	e^{-t}	$+$
$4t + 1$	$-e^{-t}$	$-$
4	e^{-t}	$+$
0	$-e^{-t}$	$-$

   Multiply diagonally and sum with the signs:

$$\begin{aligned} u(t) &= (2t^2 + t)(-e^{-t}) \;-\; (4t+1)(e^{-t}) \;+\; (4)(-e^{-t}) + C \\ &= -e^{-t}\big(2t^2 + t + 4t + 1 + 4\big) + C \\ &= -e^{-t}\big(2t^2 + 5t + 5\big) + C. \end{aligned}$$

   - Hence the particular solution is $x_p = u(t)e^t = -(2t^2 + 5t + 5) + Ce^t$. The constant C merges with the homogeneous constant.
   - General solution: $x(t) = Ce^t - (2t^2 + 5t + 5)$.

2. $\dfrac{dx}{dt} - 4tx = t^3$

- Homogeneous solution: $\dfrac{dx}{dt} - 4tx = 0 \;\Rightarrow\; \dfrac{dx}{x} = 4t\,dt \;\Rightarrow\; \ln|x| = 2t^2 + C \;\Rightarrow\; x_h = Ce^{2t^2}$.
- Particular solution: assume $x_p = u(t)e^{2t^2}$. Then $x_p' = u'e^{2t^2} + u \cdot 4te^{2t^2}$. Substituting into the original equation:

$$u'e^{2t^2} + 4tue^{2t^2} - 4t(ue^{2t^2}) = u'e^{2t^2} = t^3,$$

  so $u'(t) = t^3e^{-2t^2}$.
- Integrate $u'(t)$: let $s = t^2$, then $ds = 2t\,dt$ and $t^3dt = t^2 \cdot t\,dt = s \cdot \frac{ds}{2} = \frac{s}{2}\,ds$. Hence

$$u(t) = \int t^3e^{-2t^2}\,dt = \int \frac{s}{2}e^{-2s}\,ds = \frac{1}{2}\int se^{-2s}\,ds.$$

  Now integrate $\int se^{-2s}\,ds$ by parts (or use tabular method):

Differentiate s	Integrate e^{-2s}	Sign
s	e^{-2s}	$+$
1	$-\frac{1}{2}e^{-2s}$	$-$
0	$\frac{1}{4}e^{-2s}$	$+$

  Multiplying diagonally:

$$\int se^{-2s}\,ds = s\left(-\frac{1}{2}e^{-2s}\right) - 1\left(\frac{1}{4}e^{-2s}\right) + 0 = -\frac{s}{2}e^{-2s} - \frac{1}{4}e^{-2s} + C.$$

  Thus

$$u(t) = \frac{1}{2}\left(-\frac{s}{2}e^{-2s} - \frac{1}{4}e^{-2s}\right) + C = -\frac{s}{4}e^{-2s} - \frac{1}{8}e^{-2s} + C.$$

  Back-substitute $s = t^2$:

$$u(t) = -e^{-2t^2}\left(\frac{t^2}{4} + \frac{1}{8}\right) + C.$$

- Hence the particular solution is

$$x_p = u(t)e^{2t^2} = -\left(\frac{t^2}{4} + \frac{1}{8}\right) + Ce^{2t^2}.$$

  The constant C merges with the homogeneous constant.
- General solution: $x(t) = Ce^{2t^2} - \frac{2t^2+1}{8}$.

3. $\dfrac{dy}{dx} + 3y = 5x^2 + 2$

- Homogeneous solution: $\dfrac{dy}{dx} + 3y = 0 \;\Rightarrow\; \dfrac{dy}{y} = -3\,dx \;\Rightarrow\; \ln|y| = -3x + C \;\Rightarrow\; y_h = Ce^{-3x}$.
- Particular solution: assume $y_p = u(x)e^{-3x}$. Then $y_p' = u'e^{-3x} - 3ue^{-3x}$. Substituting into the original equation:

$$u'e^{-3x} - 3ue^{-3x} + 3ue^{-3x} = u'e^{-3x} = 5x^2 + 2,$$

  so $u'(x) = (5x^2 + 2)e^{3x}$.
- Integrate $u'(x)$ using the tabular method (differentiate the polynomial, integrate e^{3x}):

Differentiate $(5x^2 + 2)$	Integrate e^{3x}	Sign
$5x^2 + 2$	e^{3x}	$+$
$10x$	$\frac{e^{3x}}{3}$	$-$
10	$\frac{e^{3x}}{9}$	$+$
0	$\frac{e^{3x}}{27}$	$-$ (not needed)

Multiply diagonally and sum with the signs:

$$\begin{aligned}\int (5x^2+2)e^{3x}\,dx &= (5x^2+2)\left(\frac{e^{3x}}{3}\right) - (10x)\left(\frac{e^{3x}}{9}\right) + (10)\left(\frac{e^{3x}}{27}\right) + C\\ &= e^{3x}\left(\frac{5x^2+2}{3} - \frac{10x}{9} + \frac{10}{27}\right) + C\\ &= e^{3x}\left(\frac{9(5x^2+2) - 30x + 10}{27}\right) + C\\ &= e^{3x}\left(\frac{45x^2 + 18 - 30x + 10}{27}\right) + C\\ &= e^{3x}\left(\frac{45x^2 - 30x + 28}{27}\right) + C.\end{aligned}$$

- Hence
$$u(x) = e^{3x}\left(\frac{45x^2 - 30x + 28}{27}\right) + C.$$
- Then the particular solution is
$$y_p = u(x)e^{-3x} = \frac{45x^2 - 30x + 28}{27} + Ce^{-3x}.$$
The constant C merges with the homogeneous constant.
- General solution:
$$y = y_h + y_p = Ce^{-3x} + \frac{45x^2 - 30x + 28}{27}.$$
(Equivalently, $y = Ce^{-3x} + \frac{5}{3}x^2 - \frac{10}{9}x + \frac{28}{27}$.)

4. $\dfrac{dy}{dx} - 2xy = e^{x^2}\sin x$

- Homogeneous solution: $\dfrac{dy}{dx} - 2xy = 0 \Rightarrow \dfrac{dy}{y} = 2x\,dx \Rightarrow \ln|y| = x^2 + C \Rightarrow y_h = Ce^{x^2}$.
- Particular solution: assume $y_p = u(x)e^{x^2}$. Then $y_p' = u'e^{x^2} + u \cdot 2xe^{x^2}$. Substituting into the original equation:
$$u'e^{x^2} + 2xue^{x^2} - 2x(ue^{x^2}) = u'e^{x^2} = e^{x^2}\sin x,$$
so $u'(x) = \sin x$.
- Integrate $u'(x)$:
$$u(x) = \int \sin x\,dx = -\cos x + C.$$
- Hence the particular solution is
$$y_p = u(x)e^{x^2} = (-\cos x)e^{x^2} + Ce^{x^2}.$$
The constant C merges with the homogeneous constant.
- General solution:
$$y = y_h + y_p = Ce^{x^2} - e^{x^2}\cos x = e^{x^2}\big(C - \cos x\big).$$

5. $x\dfrac{dy}{dx} + 3y = \dfrac{\ln x}{x^2}$

- Rewrite in standard form: $\dfrac{dy}{dx} + \dfrac{3}{x}y = \dfrac{\ln x}{x^3}$.
- Homogeneous solution: $\dfrac{dy}{dx} + \dfrac{3}{x}y = 0 \Rightarrow \dfrac{dy}{y} = -\dfrac{3}{x}\,dx \Rightarrow \ln|y| = -3\ln|x| + C \Rightarrow y_h = Cx^{-3} = \dfrac{C}{x^3}$.

- Particular solution: assume $y_p = u(x) \cdot \frac{1}{x^3}$. Then

$$y_p' = u' \cdot \frac{1}{x^3} + u \cdot \left(-\frac{3}{x^4}\right).$$

Substitute into the original equation:

$$\left(\frac{u'}{x^3} - \frac{3u}{x^4}\right) + \frac{3}{x}\left(\frac{u}{x^3}\right) = \frac{u'}{x^3} = \frac{\ln x}{x^3},$$

so $u'(x) = \ln x$.

- Integrate $u'(x)$:

$$u(x) = \int \ln x \, dx = x \ln x - x + C.$$

- Hence the particular solution is

$$y_p = u(x) \cdot \frac{1}{x^3} = \frac{x \ln x - x}{x^3} + \frac{C}{x^3} = \frac{\ln x - 1}{x^2} + \frac{C}{x^3}.$$

The constant C merges with the homogeneous constant.

- General solution:

$$y = y_h + y_p = \frac{C}{x^3} + \frac{\ln x - 1}{x^2}.$$

6. $\dfrac{dy}{dx} + y \tan x = \sec^3 x$

- Homogeneous solution: $\dfrac{dy}{dx} + y \tan x = 0 \;\Rightarrow\; \dfrac{dy}{y} = -\tan x \, dx \;\Rightarrow\; \ln|y| = \ln|\cos x| + C \;\Rightarrow$ $y_h = C \cos x$.
- Particular solution: assume $y_p = u(x) \cos x$. Then $y_p' = u' \cos x - u \sin x$. Substituting into the original equation:

$$u' \cos x - u \sin x + (u \cos x) \tan x = u' \cos x - u \sin x + u \cos x \cdot \frac{\sin x}{\cos x} = u' \cos x = \sec^3 x,$$

so $u'(x) = \dfrac{\sec^3 x}{\cos x} = \sec^4 x$.

- Integrate $u'(x) = \sec^4 x$. Use the identity $\sec^4 x = (1 + \tan^2 x) \sec^2 x$. Let $t = \tan x$, then $dt = \sec^2 x \, dx$, so

$$u(x) = \int \sec^4 x \, dx = \int (1 + t^2) \, dt = t + \frac{t^3}{3} + C = \tan x + \frac{\tan^3 x}{3} + C.$$

- Hence the particular solution is

$$y_p = u(x) \cos x = \left(\tan x + \frac{\tan^3 x}{3}\right) \cos x + C \cos x = \sin x + \frac{\sin^3 x}{3 \cos^2 x} + C \cos x.$$

The constant C merges with the homogeneous constant.

- General solution:

$$y = y_h + y_p = C \cos x + \sin x + \frac{\sin^3 x}{3 \cos^2 x}.$$

7. $(1 + x^2)\dfrac{dy}{dx} + 2xy = 4x^3$

- Rewrite in standard form: divide both sides by $1 + x^2$:

$$\frac{dy}{dx} + \frac{2x}{1 + x^2} \, y = \frac{4x^3}{1 + x^2}.$$

- Homogeneous solution: $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2}\, y = 0$. Separate variables:

$$\frac{dy}{y} = -\frac{2x}{1+x^2}\, dx \;\Rightarrow\; \ln|y| = -\ln(1+x^2) + C \;\Rightarrow\; y_h = \frac{C}{1+x^2}.$$

- Particular solution: assume $y_p = \dfrac{u(x)}{1+x^2}$. Then differentiate:

$$y_p' = \frac{u'(1+x^2) - u\cdot 2x}{(1+x^2)^2}.$$

  Substitute into the original equation (in standard form):

$$\frac{u'(1+x^2) - 2xu}{(1+x^2)^2} + \frac{2x}{1+x^2}\cdot\frac{u}{1+x^2} = \frac{4x^3}{1+x^2}.$$

  The second term on the left is $\dfrac{2xu}{(1+x^2)^2}$, so it cancels with the $-2xu$ term:

$$\frac{u'(1+x^2)}{(1+x^2)^2} = \frac{4x^3}{1+x^2} \;\Rightarrow\; \frac{u'}{1+x^2} = \frac{4x^3}{1+x^2}.$$

  Thus $u'(x) = 4x^3$.

- Integrate $u'(x)$:

$$u(x) = \int 4x^3\, dx = x^4 + C.$$

- Hence the particular solution is

$$y_p = \frac{u(x)}{1+x^2} = \frac{x^4}{1+x^2} + \frac{C}{1+x^2}.$$

  The constant C merges with the homogeneous constant.

- General solution:

$$y = y_h + y_p = \frac{C}{1+x^2} + \frac{x^4}{1+x^2} = \frac{x^4 + C}{1+x^2}.$$

  (Note: $x^4/(1+x^2)$ can be simplified by polynomial division: $x^4 = (x^2-1)(1+x^2) + 1$, so $y = x^2 - 1 + \frac{1+C}{1+x^2}$. The constant C is arbitrary, so we may write $y = x^2 - 1 + \frac{K}{1+x^2}$.)

8. $\dfrac{dx}{dt} + \dfrac{2x}{t} = \cos t$

- Homogeneous solution: $\dfrac{dx}{dt} + \dfrac{2}{t}x = 0 \;\Rightarrow\; \dfrac{dx}{x} = -\dfrac{2}{t}\, dt \;\Rightarrow\; \ln|x| = -2\ln|t| + C \;\Rightarrow\; x_h = \dfrac{C}{t^2}$.

- Particular solution: assume $x_p = \dfrac{u(t)}{t^2}$. Then

$$x_p' = \frac{u'}{t^2} - \frac{2u}{t^3}.$$

  Substitute into the original equation:

$$\frac{u'}{t^2} - \frac{2u}{t^3} + \frac{2}{t}\cdot\frac{u}{t^2} = \frac{u'}{t^2} = \cos t,$$

  so $u'(t) = t^2 \cos t$.

- Integrate $u'(t) = t^2\cos t$ using the tabular method (differentiate t^2, integrate $\cos t$):

Differentiate t^2	Integrate $\cos t$	Sign
t^2	$\sin t$	$+$
$2t$	$-\cos t$	$-$
2	$-\sin t$	$+$
0	$\cos t$	$-$

Multiply diagonally and sum with the signs:

$$\begin{aligned} u(t) &= (t^2)(\sin t) - (2t)(-\cos t) + (2)(-\sin t) + C \\ &= t^2 \sin t + 2t\cos t - 2\sin t + C. \end{aligned}$$

- Hence the particular solution is

$$x_p = \frac{u(t)}{t^2} = \frac{t^2 \sin t + 2t\cos t - 2\sin t}{t^2} + \frac{C}{t^2} = \sin t + \frac{2\cos t}{t} - \frac{2\sin t}{t^2} + \frac{C}{t^2}.$$

- General solution:

$$x = x_h + x_p = \frac{C}{t^2} + \sin t + \frac{2\cos t}{t} - \frac{2\sin t}{t^2}.$$

9. $\dfrac{dy}{dx} + \dfrac{y}{x+1} = (x+1)^2 e^{-x}$

- Homogeneous solution: $\dfrac{dy}{dx} + \dfrac{1}{x+1}y = 0 \Rightarrow \dfrac{dy}{y} = -\dfrac{dx}{x+1} \Rightarrow \ln|y| = -\ln|x+1| + C \Rightarrow$ $y_h = \dfrac{C}{x+1}$.
- Particular solution: assume $y_p = \dfrac{u(x)}{x+1}$. Then

$$y_p' = \frac{u'}{x+1} - \frac{u}{(x+1)^2}.$$

Substitute into the original equation:

$$\frac{u'}{x+1} - \frac{u}{(x+1)^2} + \frac{1}{x+1} \cdot \frac{u}{x+1} = \frac{u'}{x+1} = (x+1)^2 e^{-x},$$

so $u'(x) = (x+1)^3 e^{-x}$.

- Integrate $u'(x) = (x+1)^3 e^{-x}$ using the tabular method (differentiate $(x+1)^3$, integrate e^{-x}):

Differentiate $(x+1)^3$	Integrate e^{-x}	Sign
$(x+1)^3$	e^{-x}	$+$
$3(x+1)^2$	$-e^{-x}$	$-$
$6(x+1)$	e^{-x}	$+$
6	$-e^{-x}$	$-$

Multiply diagonally and sum with the signs:

$$\begin{aligned} u(x) &= (x+1)^3(-e^{-x}) - 3(x+1)^2(e^{-x}) + 6(x+1)(-e^{-x}) - 6(e^{-x}) + C \\ &= -e^{-x}\big[(x+1)^3 + 3(x+1)^2 + 6(x+1) + 6\big] + C. \end{aligned}$$

- Hence the particular solution is

$$y_p = \frac{u(x)}{x+1} = -\frac{e^{-x}}{x+1}\big[(x+1)^3 + 3(x+1)^2 + 6(x+1) + 6\big] + \frac{C}{x+1}.$$

Simplify the polynomial fraction:

$$\frac{(x+1)^3 + 3(x+1)^2 + 6(x+1) + 6}{x+1} = (x+1)^2 + 3(x+1) + 6 + \frac{6}{x+1}.$$

Thus

$$y_p = -e^{-x}\left[(x+1)^2 + 3(x+1) + 6 + \frac{6}{x+1}\right] + \frac{C}{x+1}.$$

The constant C merges with the homogeneous constant.

- General solution:

$$y = y_h + y_p = \frac{C}{x+1} - e^{-x}\left[(x+1)^2 + 3(x+1) + 6 + \frac{6}{x+1}\right].$$

Equivalently, combining the terms with denominator $x+1$:

$$y = \frac{C - 6e^{-x}}{x+1} - e^{-x}\big[(x+1)^2 + 3(x+1) + 6\big].$$

10. $\dfrac{dy}{dx} + y\cot x = 2x\csc x$

- Homogeneous solution: $\dfrac{dy}{dx} + y\cot x = 0 \Rightarrow \dfrac{dy}{y} = -\cot x\,dx \Rightarrow \ln|y| = -\ln|\sin x| + C \Rightarrow$ $y_h = \dfrac{C}{\sin x} = C\csc x$.
- Particular solution: assume $y_p = u(x)\csc x$. Then $y_p' = u'\csc x - u\csc x\cot x$. Substitute into the original equation:

$$u'\csc x - u\csc x\cot x + (u\csc x)\cot x = u'\csc x = 2x\csc x,$$

so $u'(x) = 2x$.
- Integrate $u'(x)$:

$$u(x) = \int 2x\,dx = x^2 + C.$$

- Hence the particular solution is

$$y_p = u(x)\csc x = (x^2 + C)\csc x.$$

The constant C merges with the homogeneous constant.
- General solution:

$$y = y_h + y_p = C\csc x + x^2\csc x = (x^2 + C)\csc x.$$

11. $\dfrac{dy}{dx} - \dfrac{y}{x\ln x} = \dfrac{1}{x^2}$

- Homogeneous solution: $\dfrac{dy}{dx} - \dfrac{1}{x\ln x}y = 0 \Rightarrow \dfrac{dy}{y} = \dfrac{dx}{x\ln x}$. Integrate: $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x\ln x}$. Let $u = \ln x$, then $du = \frac{dx}{x}$, so $\displaystyle\int \frac{du}{u} = \ln|u| = \ln|\ln x|$. Thus $\ln|y| = \ln|\ln x| + C \Rightarrow y_h = C\ln x$.
- Particular solution: assume $y_p = u(x)\ln x$. Then

$$y_p' = u'\ln x + \frac{u}{x}.$$

Substitute into the original equation:

$$u'\ln x + \frac{u}{x} - \frac{1}{x\ln x}\cdot(u\ln x) = u'\ln x = \frac{1}{x^2},$$

so $u'(x) = \dfrac{1}{x^2\ln x}$.
- Integrate $u'(x)$:

$$u(x) = \int \frac{dx}{x^2\ln x} + C.$$

This integral cannot be expressed in elementary functions, so we leave it in integral form.
- Hence the particular solution is

$$y_p = \left(\int \frac{dx}{x^2\ln x} + C\right)\ln x.$$

The constant C merges with the homogeneous constant.

- General solution:

$$y = y_h + y_p = C\ln x + \ln x \int \frac{dx}{x^2 \ln x}.$$

12. $\dfrac{dy}{dx} + 4y = 3e^{-4x} + 2\sin 2x$

- Homogeneous solution: $\dfrac{dy}{dx} + 4y = 0 \;\Rightarrow\; \dfrac{dy}{y} = -4\,dx \;\Rightarrow\; \ln|y| = -4x + C \;\Rightarrow\; y_h = Ce^{-4x}$.
- Particular solution: assume $y_p = u(x)e^{-4x}$. Then $y_p' = u'e^{-4x} - 4ue^{-4x}$. Substituting into the original equation:

$$u'e^{-4x} - 4ue^{-4x} + 4ue^{-4x} = u'e^{-4x} = 3e^{-4x} + 2\sin 2x,$$

so $u'(x) = 3 + 2e^{4x}\sin 2x$.
- Integrate $u'(x)$:

$$u(x) = \int 3\,dx + 2\int e^{4x}\sin 2x\,dx + C.$$

The first integral is $3x$. For the second, use the formula $\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$ with $a = 4,\ b = 2$:

$$\int e^{4x}\sin 2x\,dx = \frac{e^{4x}(4\sin 2x - 2\cos 2x)}{4^2+2^2} = \frac{e^{4x}(4\sin 2x - 2\cos 2x)}{20} = \frac{e^{4x}(2\sin 2x - \cos 2x)}{10}.$$

Hence

$$u(x) = 3x + 2\cdot\frac{e^{4x}(2\sin 2x - \cos 2x)}{10} + C = 3x + \frac{e^{4x}(2\sin 2x - \cos 2x)}{5} + C.$$

- Then the particular solution is

$$y_p = u(x)e^{-4x} = e^{-4x}\left(3x + \frac{e^{4x}(2\sin 2x - \cos 2x)}{5} + C\right) = 3xe^{-4x} + \frac{2\sin 2x - \cos 2x}{5} + Ce^{-4x}.$$

The constant C merges with the homogeneous constant.
- General solution:

$$y = y_h + y_p = Ce^{-4x} + 3xe^{-4x} + \frac{2\sin 2x - \cos 2x}{5} = e^{-4x}(3x + C) + \frac{2\sin 2x - \cos 2x}{5}.$$

13. $t\dfrac{dx}{dt} + 4x = e^t$

- Rewrite in standard form: $\dfrac{dx}{dt} + \dfrac{4}{t}x = \dfrac{e^t}{t}$.
- Homogeneous solution: $\dfrac{dx}{dt} + \dfrac{4}{t}x = 0 \;\Rightarrow\; \dfrac{dx}{x} = -\dfrac{4}{t}\,dt \;\Rightarrow\; \ln|x| = -4\ln|t| + C \;\Rightarrow\; x_h = \dfrac{C}{t^4}$.
- Particular solution: assume $x_p = \dfrac{u(t)}{t^4}$. Then

$$x_p' = \frac{u'}{t^4} - \frac{4u}{t^5}.$$

Substitute into the standard form:

$$\frac{u'}{t^4} - \frac{4u}{t^5} + \frac{4}{t}\cdot\frac{u}{t^4} = \frac{u'}{t^4} = \frac{e^t}{t},$$

so $u'(t) = t^3e^t$.

- Integrate $u'(t) = t^3 e^t$ using the tabular method (differentiate t^3, integrate e^t):

Differentiate t^3	Integrate e^t	Sign
t^3	e^t	$+$
$3t^2$	e^t	$-$
$6t$	e^t	$+$
6	e^t	$-$
0	e^t	$+$

Multiply diagonally and sum with the signs:

$$\begin{aligned} u(t) &= t^3 e^t - 3t^2 e^t + 6te^t - 6e^t + C \\ &= e^t(t^3 - 3t^2 + 6t - 6) + C. \end{aligned}$$

- Hence the particular solution is

$$x_p = \frac{u(t)}{t^4} = \frac{e^t(t^3 - 3t^2 + 6t - 6)}{t^4} + \frac{C}{t^4}.$$

The constant C merges with the homogeneous constant.

- General solution:

$$x = x_h + x_p = \frac{C}{t^4} + \frac{e^t(t^3 - 3t^2 + 6t - 6)}{t^4} = \frac{C + e^t(t^3 - 3t^2 + 6t - 6)}{t^4}.$$

14. $\dfrac{dx}{dt} - (2\cot 2t)x = \cos t$

- Homogeneous solution: $\dfrac{dx}{dt} - 2\cot 2t\, x = 0 \Rightarrow \dfrac{dx}{x} = 2\cot 2t\, dt \Rightarrow \ln|x| = \ln|\sin 2t| + C \Rightarrow$ $x_h = C\sin 2t$.
- Particular solution: assume $x_p = u(t)\sin 2t$. Then

$$x_p' = u'\sin 2t + 2u\cos 2t.$$

Substitute into the original equation:

$$u'\sin 2t + 2u\cos 2t - 2\cot 2t\,(u\sin 2t) = u'\sin 2t + 2u\cos 2t - 2u\frac{\cos 2t}{\sin 2t}\sin 2t = u'\sin 2t = \cos t,$$

so $u'(t) = \dfrac{\cos t}{\sin 2t}$.

- Simplify using $\sin 2t = 2\sin t\cos t$:

$$u'(t) = \frac{\cos t}{2\sin t\cos t} = \frac{1}{2\sin t} = \frac{1}{2}\csc t.$$

- Integrate $u'(t)$:

$$u(t) = \frac{1}{2}\int \csc t\, dt = \frac{1}{2}\ln|\csc t - \cot t| + C.$$

- Hence the particular solution is

$$x_p = u(t)\sin 2t = \frac{1}{2}\sin 2t\,\ln|\csc t - \cot t| + C\sin 2t.$$

- General solution (absorbing the constant into C):

$$x = x_h + x_p = C\sin 2t + \frac{1}{2}\sin 2t\,\ln|\csc t - \cot t| = \sin 2t\left(C + \frac{1}{2}\ln|\csc t - \cot t|\right).$$

15. $\dfrac{dx}{dt} + 6t^2 x = t^2 + 2t^5$

- Homogeneous solution: $\dfrac{dx}{dt} + 6t^2x = 0 \;\Rightarrow\; \dfrac{dx}{x} = -6t^2\,dt \;\Rightarrow\; \ln|x| = -2t^3 + C \;\Rightarrow\; x_h = Ce^{-2t^3}$.
- Particular solution: assume $x_p = u(t)e^{-2t^3}$. Then

$$x_p' = u'e^{-2t^3} + u \cdot (-6t^2)e^{-2t^3}.$$

Substitute into the original equation:

$$u'e^{-2t^3} - 6t^2ue^{-2t^3} + 6t^2ue^{-2t^3} = u'e^{-2t^3} = t^2 + 2t^5,$$

so $u'(t) = (t^2 + 2t^5)e^{2t^3}$.
- Integrate $u'(t)$. Let $w = t^3$, then $dw = 3t^2\,dt$ and $t^2\,dt = dw/3$. Also $t^5 = t^3 \cdot t^2 = w \cdot t^2$. Then

$$u(t) = \int (t^2 + 2t^5)e^{2t^3}\,dt = \int t^2e^{2t^3}\,dt + 2\int t^5e^{2t^3}\,dt.$$

In terms of w:

$$\int t^2e^{2t^3}\,dt = \int e^{2w}\frac{dw}{3} = \frac{1}{3}\int e^{2w}\,dw = \frac{1}{6}e^{2w},$$

$$\int t^5e^{2t^3}\,dt = \int we^{2w}\frac{dw}{3} = \frac{1}{3}\int we^{2w}\,dw.$$

Using integration by parts or the tabular method:

$$\int we^{2w}\,dw = \frac{w}{2}e^{2w} - \frac{1}{4}e^{2w} + \text{constant}.$$

Thus

$$2\int t^5e^{2t^3}\,dt = 2 \cdot \frac{1}{3}\left(\frac{w}{2}e^{2w} - \frac{1}{4}e^{2w}\right) = \frac{w}{3}e^{2w} - \frac{1}{6}e^{2w}.$$

Combining with the first term:

$$u(t) = \frac{1}{6}e^{2w} + \frac{w}{3}e^{2w} - \frac{1}{6}e^{2w} + C = \frac{w}{3}e^{2w} + C = \frac{t^3}{3}e^{2t^3} + C.$$

- Hence the particular solution is

$$x_p = u(t)e^{-2t^3} = \frac{t^3}{3} + Ce^{-2t^3}.$$

The constant C merges with the homogeneous constant.
- General solution:

$$x = x_h + x_p = Ce^{-2t^3} + \frac{t^3}{3}.$$

Exercise 08: Solutions of Linear First-Order Differential Equations -Integrating Factor (Second Method)

1. $\frac{dx}{dt} - x = t + 2t^2$
   - Compute the integrating factor: $\mu(t) = e^{\int -1dt} = e^{-t}$.
   - Multiply the equation by $\mu(t)$: $e^{-t}\frac{dx}{dt} - e^{-t}x = (t + 2t^2)e^{-t}$.
   - This simplifies to: $\frac{d}{dt}(e^{-t}x) = (t + 2t^2)e^{-t}$.
   - Integrate: $e^{-t}x = \int(t + 2t^2)e^{-t}dt + C = -(2t^2 + 5t + 5)e^{-t} + C$.
   - Solve for x: $x = -(2t^2 + 5t + 5) + Ce^t$.
2. $\frac{dx}{dt} - 4tx = t^3$
   - Compute the integrating factor: $\mu(t) = e^{\int -4tdt} = e^{-2t^2}$.

- Multiply the equation by $\mu(t)$: $e^{-2t^2}\frac{dx}{dt} - 4te^{-2t^2}x = t^3e^{-2t^2}$.
- This simplifies to: $\frac{d}{dt}(e^{-2t^2}x) = t^3e^{-2t^2}$.
- Integrate: $e^{-2t^2}x = \int t^3e^{-2t^2}dt + C = -\frac{1}{8}(2t^2+1)e^{-2t^2} + C$.
- Solve for x: $x = -\frac{1}{4}t^2 - \frac{1}{8} + Ce^{2t^2}$.

3. $\frac{dy}{dx} + 3y = 5x^2 + 2$
   - Compute the integrating factor: $\mu(x) = e^{\int 3\,dx} = e^{3x}$.
   - Multiply and simplify: $\frac{d}{dx}(e^{3x}y) = (5x^2+2)e^{3x}$.
   - Integrate:
   $$e^{3x}y = \int (5x^2+2)e^{3x}\,dx + C = \frac{1}{27}e^{3x}(45x^2 - 30x + 28) + C.$$
   - Solve for y:
   $$y = \frac{45x^2 - 30x + 28}{27} + Ce^{-3x}.$$

4. $\frac{dy}{dx} - 2xy = e^{x^2}\sin x$
   - Compute the integrating factor: $\mu(x) = e^{\int -2x dx} = e^{-x^2}$.
   - Multiply and simplify: $\frac{d}{dx}(e^{-x^2}y) = \sin x$.
   - Integrate: $e^{-x^2}y = -\cos x + C$.
   - Solve for y: $y = e^{x^2}(C - \cos x)$.

5. $x\frac{dy}{dx} + 3y = \frac{\ln x}{x^2} \implies \frac{dy}{dx} + \frac{3}{x}y = \frac{\ln x}{x^3}$
   - Compute the integrating factor: $\mu(x) = e^{\int \frac{3}{x}dx} = x^3$.
   - Multiply and simplify: $\frac{d}{dx}(x^3y) = \ln x$.
   - Integrate: $x^3y = x\ln x - x + C$.
   - Solve for y: $y = \frac{\ln x - 1}{x^2} + \frac{C}{x^3}$.

6. $\frac{dy}{dx} + y\tan x = \sec^3 x$
   - Compute the integrating factor: $\mu(x) = e^{\int \tan x dx} = \sec x$.
   - Multiply and simplify: $\frac{d}{dx}(y\sec x) = \sec^4 x$.
   - Integrate: $y\sec x = \tan x + \frac{1}{3}\tan^3 x + C$.
   - Solve for y: $y = \sin x + \frac{1}{3}\sin x\tan^2 x + C\cos x$.

7. $(1+x^2)\frac{dy}{dx} + 2xy = 4x^3 \implies \frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^3}{1+x^2}$
   - Compute $\mu(x) = e^{\int \frac{2x}{1+x^2}dx} = 1 + x^2$.
   - Simplify: $\frac{d}{dx}((1+x^2)y) = 4x^3$.
   - Integrate: $(1+x^2)y = x^4 + C$.
   - Solve for y: $y = \frac{x^4+C}{1+x^2}$.

8. $\frac{dx}{dt} + \frac{2x}{t} = \cos t$
   - Compute $\mu(t) = e^{\int \frac{2}{t}dt} = t^2$.
   - Simplify: $\frac{d}{dt}(t^2x) = t^2\cos t$.
   - Integrate: $t^2x = t^2\sin t + 2t\cos t - 2\sin t + C$.
   - Solve for x: $x = \sin t + \frac{2\cos t}{t} - \frac{2\sin t}{t^2} + \frac{C}{t^2}$.

9. $\frac{dy}{dx} + \frac{y}{x+1} = (x+1)^2e^{-x}$

- Compute $\mu(x) = e^{\int \frac{1}{x+1}\,dx} = x+1$.
- Simplify: $\frac{d}{dx}((x+1)y) = (x+1)^3 e^{-x}$.
- Integrate: $(x+1)y = -(x^3+6x^2+15x+16)e^{-x} + C$.
- Solve for y: $y = \frac{C-(x^3+6x^2+15x+16)e^{-x}}{x+1}$.

10. $\frac{dy}{dx} + y\cot x = 2x\csc x$

- Compute $\mu(x) = e^{\int \cot x dx} = \sin x$.
- Simplify: $\frac{d}{dx}(y\sin x) = 2x$.
- Integrate: $y\sin x = x^2 + C$.
- Solve for y: $y = (x^2 + C)\csc x$.

11. $\frac{dy}{dx} - \frac{y}{x\ln x} = \frac{1}{x^2}$

- Compute $\mu(x) = e^{\int -\frac{1}{x\ln x}\,dx} = \frac{1}{\ln x}$.
- Simplify: $\frac{d}{dx}(\frac{y}{\ln x}) = \frac{1}{x^2\ln x}$.
- Integrate: $\frac{y}{\ln x} = \int \frac{1}{x^2\ln x}dx + C$.
- Solve for y: $y = \ln x(\int \frac{1}{x^2\ln x}dx + C)$.

12. $\frac{dy}{dx} + 4y = 3e^{-4x} + 2\sin 2x$

- Compute the integrating factor: $\mu(x) = e^{\int 4\,dx} = e^{4x}$.
- Multiply the equation by $\mu(x)$: $e^{4x}\frac{dy}{dx} + 4e^{4x}y = 3 + 2e^{4x}\sin 2x$.
- This simplifies to: $\frac{d}{dx}\left(e^{4x}y\right) = 3 + 2e^{4x}\sin 2x$.
- Integrate:

$$e^{4x}y = \int 3\,dx + 2\int e^{4x}\sin 2x\,dx + C = 3x + \frac{1}{5}e^{4x}(2\sin 2x - \cos 2x) + C.$$

- Solve for y:

$$y = 3xe^{-4x} + \frac{1}{5}(2\sin 2x - \cos 2x) + Ce^{-4x}.$$

13. $t\frac{dx}{dt} + 4x = e^t \implies \frac{dx}{dt} + \frac{4}{t}x = \frac{e^t}{t}$

- Compute $\mu(t) = t^4$.
- Simplify: $\frac{d}{dt}(xt^4) = t^3e^t$.
- Integrate: $xt^4 = (t^3 - 3t^2 + 6t - 6)e^t + C$.
- Solve for x: $x = \frac{(t^3-3t^2+6t-6)e^t+C}{t^4}$.

14. $\frac{dx}{dt} - (2\cot 2t)x = \cos t$

- Compute $\mu(t) = e^{\int -2\cot 2t dt} = \frac{1}{\sin 2t}$.
- Simplify: $\frac{d}{dt}(\frac{x}{\sin 2t}) = \frac{\cos t}{2\sin t\cos t} = \frac{1}{2\sin t}$.
- Integrate: $\frac{x}{\sin 2t} = \frac{1}{2}\ln|\csc t - \cot t| + C$.
- Solve for x: $x = \sin 2t(\frac{1}{2}\ln|\tan(t/2)| + C)$.

15. $\frac{dx}{dt} + 6t^2x = t^2 + 2t^5$

- Compute $\mu(t) = e^{2t^3}$.
- Simplify: $\frac{d}{dt}(xe^{2t^3}) = (t^2 + 2t^5)e^{2t^3}$.
- Integrate: $xe^{2t^3} = \frac{1}{6}e^{2t^3} + \frac{1}{3}t^3e^{2t^3} - \frac{1}{6}e^{2t^3} + C = \frac{1}{3}t^3e^{2t^3} + C$.
- Solve for x: $x = \frac{1}{3}t^3 + Ce^{-2t^3}$.

Exercise 09: Solution of a Differential Equation Reducible to Homogeneous Form

1. $(3y - 7x + 7)\,dx - (3x - 7y - 3)\,dy = 0$

   - Rewrite as $\frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3}$. Check $ae - bd = (-7)(-7) - 3 \cdot 3 = 40 \neq 0 \to$ intersecting lines.
   - Find intersection: solve $\begin{cases} -7x + 3y + 7 = 0 \\ 3x - 7y - 3 = 0 \end{cases} \to (x_0, y_0) = (1, 0)$.
   - Translate: $X = x - 1$, $Y = y \to \frac{dY}{dX} = \frac{3Y-7X}{3X-7Y}$ (homogeneous).
   - Substitute $Y = vX$: $v + X\frac{dv}{dX} = \frac{3v-7}{3-7v} \to X\frac{dv}{dX} = \frac{7(v^2-1)}{3-7v}$.
   - Separate variables: $\frac{3-7v}{7(v^2-1)}\,dv = \frac{dX}{X}$. Use partial fractions $\frac{3-7v}{v^2-1} = -\frac{2}{v-1} - \frac{5}{v+1}$.
   - Integrate: $-\frac{2}{7}\ln|v-1| - \frac{5}{7}\ln|v+1| = \ln|X| + C \to X^7(v-1)^2(v+1)^5 = C_1$.
   - Back-substitute $v = Y/X$, $X = x - 1$, $Y = y$ and simplify:

   $$(x-1)^7\left(\frac{y}{x-1} - 1\right)^2\left(\frac{y}{x-1} + 1\right)^5 = C_1$$

   $\to (y - x + 1)^2(x + y - 1)^5 = C$.

2. $(x + 2y + 1)\,dx - (2x + 4y + 3)\,dy = 0$

   - Rewrite as $\frac{dy}{dx} = \frac{x+2y+1}{2x+4y+3}$. Compute $ae - bd = 1 \cdot 4 - 2 \cdot 2 = 0 \to$ parallel lines.
   - Let $u = x + 2y$. Then $\frac{du}{dx} = 1 + 2\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{2}\left(\frac{du}{dx} - 1\right)$.
   - Substitute: $\frac{1}{2}\left(\frac{du}{dx} - 1\right) = \frac{u+1}{2u+3} \implies \frac{du}{dx} = 1 + \frac{2(u+1)}{2u+3} = \frac{4u+5}{2u+3}$.
   - Separate variables: $\frac{2u+3}{4u+5}\,du = dx$.
   - Integrate: $\int \frac{2u+3}{4u+5}\,du = \int dx$. Note $\frac{2u+3}{4u+5} = \frac{1}{2} + \frac{1}{2(4u+5)}$, so $\int = \frac{1}{2}u + \frac{1}{8}\ln|4u+5|$.
   - Thus $\frac{1}{2}u + \frac{1}{8}\ln|4u+5| = x + C$.
   - Back-substitute $u = x + 2y$: $\frac{1}{2}(x+2y) + \frac{1}{8}\ln|4(x+2y)+5| = x + C$.
   - Multiply by 8: $4(x+2y) + \ln|4x+8y+5| = 8x + C' \implies 4x + 8y + \ln|4x+8y+5| = 8x + C'$ $\implies 8y - 4x + \ln|4x+8y+5| = C'$.
   - Final implicit solution: $\boxed{8y - 4x + \ln|4x+8y+5| = C}$.

3. $(x + 2y + 1)\,dx - (2x - 3)\,dy = 0$

   - Rewrite as $\frac{dy}{dx} = \frac{x+2y+1}{2x-3}$. Compute $ae - bd = 1 \cdot 0 - 2 \cdot 2 = -4 \neq 0 \to$ intersecting lines.
   - Find intersection: solve $\begin{cases} x + 2y + 1 = 0 \\ 2x - 3 = 0 \end{cases} \to x_0 = \frac{3}{2}$, $y_0 = -\frac{5}{4}$.
   - Translate: $X = x - \frac{3}{2}$, $Y = y + \frac{5}{4} \to \frac{dY}{dX} = \frac{X+2Y}{2X}$, which is homogeneous.
   - Substitute $Y = vX$: $v + X\frac{dv}{dX} = \frac{1+2v}{2} = \frac{1}{2} + v \implies X\frac{dv}{dX} = \frac{1}{2}$.
   - Separate variables: $dv = \frac{1}{2X}\,dX \implies$ integrate: $v = \frac{1}{2}\ln|X| + C$.
   - Back-substitute $v = Y/X$: $Y = X\left(\frac{1}{2}\ln|X| + C\right)$.
   - Return to x, y: $y + \frac{5}{4} = \left(x - \frac{3}{2}\right)\left(\frac{1}{2}\ln\left|x - \frac{3}{2}\right| + C\right)$.
   - Final solution: $\boxed{y = \left(x - \frac{3}{2}\right)\left(\frac{1}{2}\ln\left|x - \frac{3}{2}\right| + C\right) - \frac{5}{4}}$.

4. $(x + y - 2)\,dx + (x - y + 4)\,dy = 0$

   - Rewrite as $\frac{dy}{dx} = -\frac{x+y-2}{x-y+4}$. Here $a = -1$, $b = -1$, $c = 2$, $d = 1$, $e = -1$, $f = 4$. Compute $ae - bd = (-1)(-1) - (-1)(1) = 1 + 1 = 2 \neq 0 \to$ intersecting lines.

- Find intersection: solve $\begin{cases} x+y-2=0 \\ x-y+4=0 \end{cases} \to x_0=-1,\ y_0=3.$
- Translate: $X=x+1$, $Y=y-3 \to \frac{dY}{dX}=-\frac{X+Y}{X-Y}$, which is homogeneous.
- Substitute $Y=vX$: $v+X\frac{dv}{dX}=-\frac{1+v}{1-v} \implies X\frac{dv}{dX}=-\frac{1+v}{1-v}-v=\frac{v^2-2v-1}{1-v}$.
- Separate variables: $\frac{1-v}{v^2-2v-1}\,dv=\frac{dX}{X}$.
- Integrate: note that $\frac{d}{dv}(v^2-2v-1)=2v-2=-2(1-v)$, so $\int\frac{1-v}{v^2-2v-1}\,dv=-\frac{1}{2}\ln|v^2-2v-1|$. Thus $-\frac{1}{2}\ln|v^2-2v-1|=\ln|X|+C \implies \ln|v^2-2v-1|=-2\ln|X|+C' \implies v^2-2v-1=KX^{-2}$.
- Back-substitute $v=Y/X$: $\frac{Y^2}{X^2}-2\frac{Y}{X}-1=\frac{K}{X^2} \implies$ multiply by X^2: $Y^2-2XY-X^2=K$.
- Return to x,y: $X=x+1$, $Y=y-3 \implies (y-3)^2-2(x+1)(y-3)-(x+1)^2=C$.
- Final implicit solution: $\boxed{(y-3)^2-2(x+1)(y-3)-(x+1)^2=C}$.

5. $(2x-3y+4)\,dx+(3x-2y+1)\,dy=0$

- Rewrite as $\frac{dy}{dx}=-\frac{2x-3y+4}{3x-2y+1}=\frac{-2x+3y-4}{3x-2y+1}$. Here $a=-2$, $b=3$, $c=-4$, $d=3$, $e=-2$, $f=1$. Compute $ae-bd=(-2)(-2)-(3)(3)=4-9=-5\neq 0 \implies$ intersecting lines.
- Find intersection: solve $\begin{cases} -2x+3y-4=0 \\ 3x-2y+1=0 \end{cases} \implies (x_0,y_0)=(1,2).$
- Translate: $X=x-1$, $Y=y-2 \implies \frac{dY}{dX}=\frac{-2X+3Y}{3X-2Y}$, which is homogeneous.
- Substitute $Y=vX$: $v+X\frac{dv}{dX}=\frac{-2+3v}{3-2v} \implies X\frac{dv}{dX}=\frac{-2+3v}{3-2v}-v=\frac{2(v^2-1)}{3-2v}$.
- Separate variables: $\frac{3-2v}{v^2-1}\,dv=\frac{2}{X}\,dX$.
- Use partial fractions: $\frac{3-2v}{v^2-1}=\frac{1}{2(v-1)}-\frac{5}{2(v+1)}$.
- Integrate: $\int\left(\frac{1}{2(v-1)}-\frac{5}{2(v+1)}\right)dv=2\int\frac{dX}{X} \implies \frac{1}{2}\ln|v-1|-\frac{5}{2}\ln|v+1|=2\ln|X|+C$.
- Multiply by 2: $\ln|v-1|-5\ln|v+1|=4\ln|X|+C_1 \implies \ln\left|\frac{v-1}{(v+1)^5}\right|=\ln|X|^4+C_1 \implies \frac{v-1}{(v+1)^5}=KX^4$, where $K=\pm e^{C_1}$.
- Back-substitute $v=Y/X$: $\frac{Y/X-1}{(Y/X+1)^5}=KX^4 \implies \frac{Y-X}{X}\cdot\frac{X^5}{(Y+X)^5}=KX^4 \implies \frac{Y-X}{(Y+X)^5}=K$.
- Return to x,y: $X=x-1$, $Y=y-2 \implies \frac{(y-2)-(x-1)}{[(y-2)+(x-1)]^5}=K \implies \frac{y-x-1}{(x+y-3)^5}=K$.
- Final implicit solution: $\boxed{y-x-1=C\,(x+y-3)^5}$ (with C an arbitrary constant).

6. $(x-2y+5)\,dx-(2x-y+4)\,dy=0$

- Rewrite as $\frac{dy}{dx}=\frac{x-2y+5}{2x-y+4}$. Here $a=1$, $b=-2$, $c=5$, $d=2$, $e=-1$, $f=4$. Compute $ae-bd=1\cdot(-1)-(-2)\cdot 2=-1+4=3\neq 0 \implies$ intersecting lines.
- Find intersection: solve $\begin{cases} x-2y+5=0 \\ 2x-y+4=0 \end{cases} \implies (x_0,y_0)=(-1,2).$
- Translate: $X=x+1$, $Y=y-2 \implies \frac{dY}{dX}=\frac{X-2Y}{2X-Y}$, which is homogeneous.
- Substitute $Y=vX$: $v+X\frac{dv}{dX}=\frac{1-2v}{2-v} \implies X\frac{dv}{dX}=\frac{1-2v}{2-v}-v=\frac{v^2-4v+1}{2-v}$.
- Separate variables: $\frac{2-v}{v^2-4v+1}\,dv=\frac{dX}{X}$. Note $2-v=-(v-2)$.
- Let $u=v^2-4v+1$, then $du=(2v-4)dv=2(v-2)dv \implies (v-2)dv=\frac{du}{2}$. Thus $\frac{2-v}{v^2-4v+1}dv=-\frac{(v-2)}{v^2-4v+1}dv=-\frac{1}{2}\frac{du}{u}$.
- Integrate: $-\frac{1}{2}\ln|u|=\ln|X|+C \implies \ln|v^2-4v+1|=-2\ln|X|+C_1 \implies v^2-4v+1=KX^{-2}$.
- Back-substitute $v=Y/X$: $\frac{Y^2}{X^2}-4\frac{Y}{X}+1=\frac{K}{X^2} \implies$ multiply by X^2: $Y^2-4XY+X^2=K$.
- Return to x,y: $X=x+1$, $Y=y-2 \implies (y-2)^2-4(x+1)(y-2)+(x+1)^2=C$.

- Final implicit solution: $\boxed{(y-2)^2 - 4(x+1)(y-2) + (x+1)^2 = C}$.

7. $(x+y+1)\,dx + (2x+2y+1)\,dy = 0$

   - Rewrite as $\frac{dy}{dx} = -\frac{x+y+1}{2x+2y+1}$. Here $a=1,\ b=1,\ c=1,\ d=2,\ e=2,\ f=1$. Compute $ae-bd = 1\cdot 2 - 1\cdot 2 = 0 \implies$ parallel lines.
   - Let $u = x+y \implies \frac{du}{dx} = 1 + \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{du}{dx} - 1$.
   - Substitute: $\frac{du}{dx} - 1 = -\frac{u+1}{2u+1} \implies \frac{du}{dx} = 1 - \frac{u+1}{2u+1} = \frac{2u+1-u-1}{2u+1} = \frac{u}{2u+1}$.
   - Separate variables: $\frac{2u+1}{u}\,du = dx \implies \left(2 + \frac{1}{u}\right)du = dx$.
   - Integrate: $2u + \ln|u| = x + C$.
   - Back-substitute $u = x+y$: $2(x+y) + \ln|x+y| = x + C \implies 2x+2y+\ln|x+y| = x+C \implies x + 2y + \ln|x+y| = C$.
   - Final implicit solution: $\boxed{x + 2y + \ln|x+y| = C}$.

8. $(3x+2y-5)\,dx - (2x+3y-5)\,dy = 0$

   - Rewrite as $\frac{dy}{dx} = \frac{3x+2y-5}{2x+3y-5}$. Here $a=3,\ b=2,\ c=-5,\ d=2,\ e=3,\ f=-5$. Compute $ae-bd = 3\cdot 3 - 2\cdot 2 = 9-4 = 5 \neq 0 \implies$ intersecting lines.
   - Find intersection: solve $\begin{cases} 3x+2y-5=0 \\ 2x+3y-5=0 \end{cases} \implies (x_0, y_0) = (1,1)$.
   - Translate: $X = x-1$, $Y = y-1 \implies \frac{dY}{dX} = \frac{3X+2Y}{2X+3Y}$, which is homogeneous.
   - Substitute $Y = vX$: $v + X\frac{dv}{dX} = \frac{3+2v}{2+3v} \implies X\frac{dv}{dX} = \frac{3+2v}{2+3v} - v = \frac{3(1-v^2)}{2+3v}$.
   - Separate variables: $\frac{2+3v}{3(1-v^2)}\,dv = \frac{dX}{X}$.
   - Use partial fractions: $\frac{2+3v}{1-v^2} = \frac{5/2}{1-v} - \frac{1/2}{1+v} \implies \frac{2+3v}{3(1-v^2)} = \frac{5}{6(1-v)} - \frac{1}{6(1+v)}$.
   - Integrate: $\int \left(\frac{5}{6(1-v)} - \frac{1}{6(1+v)}\right) dv = \int \frac{dX}{X} \implies -\frac{5}{6}\ln|1-v| - \frac{1}{6}\ln|1+v| = \ln|X| + C$.
   - Multiply by 6: $-5\ln|1-v| - \ln|1+v| = 6\ln|X| + C_1 \implies \ln\left(|1-v|^{-5}|1+v|^{-1}\right) = \ln|X|^6 + C_1 \implies |X|^6|1-v|^5|1+v| = K$.
   - Back-substitute $v = Y/X$: $|X|^6\left|\frac{X-Y}{X}\right|^5\left|\frac{X+Y}{X}\right| = K \implies |X|^6\frac{|X-Y|^5}{|X|^5}\frac{|X+Y|}{|X|} = |X-Y|^5|X+Y| = K$.
   - Return to x, y: $X = x-1$, $Y = y-1 \implies |(x-1)-(y-1)|^5|(x-1)+(y-1)| = K \implies |x-y|^5|x+y-2| = K$.
   - Final implicit solution (absorbing sign into constant): $\boxed{(x-y)^5(x+y-2) = C}$.

9. $(4x-y+7)\,dx + (x+3y-2)\,dy = 0$

   - Rewrite as $\frac{dy}{dx} = -\frac{4x-y+7}{x+3y-2} = \frac{-4x+y-7}{x+3y-2}$. Here $a=-4,\ b=1,\ c=-7,\ d=1,\ e=3,\ f=-2$. Compute $ae-bd = (-4)\cdot 3 - 1\cdot 1 = -12-1 = -13 \neq 0 \implies$ intersecting lines.
   - Find intersection: solve $\begin{cases} -4x+y-7=0 \\ x+3y-2=0 \end{cases} \implies (x_0, y_0) = \left(-\frac{19}{13}, \frac{15}{13}\right)$.
   - Translate: $X = x + \frac{19}{13}$, $Y = y - \frac{15}{13} \implies \frac{dY}{dX} = \frac{-4X+Y}{X+3Y}$, which is homogeneous.
   - Substitute $Y = vX$: $v + X\frac{dv}{dX} = \frac{-4+v}{1+3v} \implies X\frac{dv}{dX} = \frac{v-4}{1+3v} - v = -\frac{3v^2+4}{1+3v}$.
   - Separate variables: $\frac{1+3v}{3v^2+4}\,dv = -\frac{dX}{X}$.
   - Integrate left side: $\int \frac{1+3v}{3v^2+4}\,dv = \frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}\,v}{2}\right) + \frac{1}{2}\ln(3v^2+4)$.
   - Thus $\frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}\,v}{2}\right) + \frac{1}{2}\ln(3v^2+4) = -\ln|X| + C$.
   - Multiply by 2: $\frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}\,v}{2}\right) + \ln(3v^2+4) = -2\ln|X| + C_1$.

- Back-substitute $v = Y/X$: $\ln\left(3\frac{Y^2}{X^2}+4\right) + \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}\,Y}{2X}\right) = -2\ln|X| + C_1 \implies \ln(3Y^2 + 4X^2) + \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}\,Y}{2X}\right) = C_1$.
- Return to x, y: $X = x + \frac{19}{13}$, $Y = y - \frac{15}{13} \implies \ln\left(3\left(y-\frac{15}{13}\right)^2 + 4\left(x+\frac{19}{13}\right)^2\right) + \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}\,(y-\frac{15}{13})}{2(x+\frac{19}{13})}\right) = C$.
- Final implicit solution: $\boxed{\ln\left(3\left(y-\frac{15}{13}\right)^2 + 4\left(x+\frac{19}{13}\right)^2\right) + \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}\,(y-\frac{15}{13})}{2(x+\frac{19}{13})}\right) = C}$.

10. $(x-y)\,dx + (x+y+2)\,dy = 0$
    - Rewrite as $\frac{dy}{dx} = -\frac{x-y}{x+y+2} = \frac{-x+y}{x+y+2}$. Here $a=-1$, $b=1$, $c=0$, $d=1$, $e=1$, $f=2$. Compute $ae - bd = (-1)\cdot 1 - 1\cdot 1 = -2 \neq 0 \implies$ intersecting lines.
    - Find intersection: solve $\begin{cases} -x+y=0 \\ x+y+2=0 \end{cases} \implies (x_0, y_0) = (-1,-1)$.
    - Translate: $X = x+1$, $Y = y+1 \implies \frac{dY}{dX} = \frac{-X+Y}{X+Y} = \frac{Y-X}{X+Y}$, which is homogeneous.
    - Substitute $Y = vX$: $v + X\frac{dv}{dX} = \frac{v-1}{v+1} \implies X\frac{dv}{dX} = \frac{v-1}{v+1} - v = -\frac{1+v^2}{v+1}$.
    - Separate variables: $\frac{v+1}{1+v^2}\,dv = -\frac{dX}{X} \implies \left(\frac{v}{1+v^2} + \frac{1}{1+v^2}\right)dv = -\frac{dX}{X}$.
    - Integrate: $\frac{1}{2}\ln(1+v^2) + \arctan v = -\ln|X| + C$.
    - Multiply by 2: $\ln(1+v^2) + 2\arctan v = -2\ln|X| + C_1$.
    - Back-substitute $v = Y/X$: $\ln\left(1+\frac{Y^2}{X^2}\right) + 2\arctan\left(\frac{Y}{X}\right) = -2\ln|X| + C_1 \implies \ln\left(\frac{X^2+Y^2}{X^2}\right) + 2\arctan\left(\frac{Y}{X}\right) = -2\ln|X| + C_1 \implies \ln(X^2+Y^2) - 2\ln|X| + 2\arctan\left(\frac{Y}{X}\right) = -2\ln|X| + C_1 \implies \ln(X^2+Y^2) + 2\arctan\left(\frac{Y}{X}\right) = C_1$.
    - Return to x, y: $X = x+1$, $Y = y+1 \implies \boxed{\ln\left((x+1)^2 + (y+1)^2\right) + 2\arctan\left(\frac{y+1}{x+1}\right) = C}$.

11. $(x+y)\,dx + (x-y+2)\,dy = 0$
    - $\frac{dy}{dx} = -\frac{x+y}{x-y+2}$. Here $a=-1$, $b=-1$, $c=0$, $d=1$, $e=-1$, $f=2$. Compute $ae - bd = (-1)(-1) - (-1)(1) = 1+1 = 2 \neq 0 \implies$ intersecting lines.
    - Intersection: $\begin{cases} x+y=0 \\ x-y+2=0 \end{cases} \implies (x_0, y_0) = (-1, 1)$.
    - Translate: $X = x+1$, $Y = y-1 \implies \frac{dY}{dX} = -\frac{X+Y}{X-Y}$, homogeneous.
    - $Y = vX$: $v + X\frac{dv}{dX} = -\frac{1+v}{1-v} \implies X\frac{dv}{dX} = -\frac{1+v}{1-v} - v = -\frac{1+v^2}{1-v}$.
    - Separate: $\frac{1-v}{1+v^2}\,dv = -\frac{dX}{X} \implies \left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right)dv = -\frac{dX}{X}$.
    - Integrate: $\arctan v - \frac{1}{2}\ln(1+v^2) = -\ln|X| + C$.
    - Multiply by 2: $2\arctan v - \ln(1+v^2) = -2\ln|X| + C_1 \implies \ln\frac{1+v^2}{X^2} = 2\arctan v - C_1 \implies X^2 e^{2\arctan v} = K(1+v^2)$.
    - Back-substitute $v = Y/X$: $X^2 e^{2\arctan(Y/X)} = K(1 + Y^2/X^2)X^2 \implies e^{2\arctan(Y/X)} = K(1 + Y^2/X^2) \implies e^{2\arctan(Y/X)} = \frac{K}{X^2}(X^2+Y^2)$.
    - Return to x, y: $X = x+1$, $Y = y-1 \implies e^{2\arctan\frac{y-1}{x+1}} = \frac{K}{(x+1)^2}\left((x+1)^2 + (y-1)^2\right)$.
    - Rearranged: $\boxed{(x+1)^2 + (y-1)^2 = Ce^{-2\arctan\frac{y-1}{x+1}}}$.

12. $(2x+y-3)\,dx - (4x+2y+5)\,dy = 0$
    - Rewrite as $\frac{dy}{dx} = \frac{2x+y-3}{4x+2y+5}$. Here $a=2$, $b=1$, $c=-3$, $d=4$, $e=2$, $f=5$. Compute $ae - bd = 2\cdot 2 - 1\cdot 4 = 0 \implies$ parallel lines.
    - Let $u = 2x+y \implies \frac{du}{dx} = 2 + \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{du}{dx} - 2$.

- Substitute: $\frac{du}{dx} - 2 = \frac{u-3}{2u+5} \implies \frac{du}{dx} = \frac{u-3}{2u+5} + 2 = \frac{u-3+2(2u+5)}{2u+5} = \frac{5u+7}{2u+5}$.
- Separate variables: $\frac{2u+5}{5u+7}\, du = dx$.
- Perform division: $\frac{2u+5}{5u+7} = \frac{2}{5} + \frac{11}{5(5u+7)}$.
- Integrate: $\int \left(\frac{2}{5} + \frac{11}{5(5u+7)} \right) du = \int dx \implies \frac{2}{5}u + \frac{11}{25} \ln|5u+7| = x + C$.
- Multiply by 25: $10u + 11 \ln|5u+7| = 25x + C'$.
- Back-substitute $u = 2x+y$: $10(2x+y) + 11\ln|5(2x+y)+7| = 25x + C' \implies 20x + 10y + 11 \ln|10x+5y+7| = 25x + C' \implies 10y - 5x + 11\ln|10x+5y+7| = C'$.
- Final implicit solution: $\boxed{10y - 5x + 11\ln|10x+5y+7| = C}$.

13. $(2x+3y-1)\, dx + (4x+6y+2)\, dy = 0$

   - $\frac{dy}{dx} = -\frac{2x+3y-1}{4x+6y+2}$. Here $a = -2$, $b = -3$, $c = 1$, $d = -4$, $e = -6$, $f = -2$. Compute $ae - bd = (-2)(-6) - (-3)(-4) = 12 - 12 = 0 \implies$ parallel.
   - Let $u = 2x + 3y \implies \frac{du}{dx} = 2 + 3\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{3}\left(\frac{du}{dx} - 2\right)$.
   - Substitute: $\frac{1}{3}\left(\frac{du}{dx} - 2\right) = -\frac{u-1}{2u+2}$ (since $4x+6y+2 = 2u+2$). Multiply by 3: $\frac{du}{dx} - 2 = -\frac{3(u-1)}{2(u+1)}$ $\implies \frac{du}{dx} = 2 - \frac{3(u-1)}{2(u+1)} = \frac{4(u+1)-3(u-1)}{2(u+1)} = \frac{u+7}{2(u+1)}$.
   - Separate: $\frac{2(u+1)}{u+7}\, du = dx$. Perform division: $\frac{2u+2}{u+7} = 2 - \frac{12}{u+7}$.
   - Integrate: $2u - 12\ln|u+7| = x + C$.
   - Back-substitute $u = 2x+3y$: $2(2x+3y) - 12\ln|2x+3y+7| = x + C \implies 4x + 6y - 12\ln|2x + 3y+7| = x + C \implies 3x + 6y - 12\ln|2x+3y+7| = C$.
   - Final: $\boxed{x + 2y - 4\ln|2x+3y+7| = C}$ (dividing by 3).

14. $(3x - 5y + 1)\, dx + (x + y + 2)\, dy = 0$

   - Rewrite as $\frac{dy}{dx} = -\frac{3x-5y+1}{x+y+2} = \frac{-3x+5y-1}{x+y+2}$. Here $a = -3$, $b = 5$, $c = -1$, $d = 1$, $e = 1$, $f = 2$. Compute $ae - bd = (-3)\cdot 1 - 5 \cdot 1 = -8 \neq 0 \implies$ intersecting lines.
   - Find intersection: solve $\begin{cases} -3x + 5y - 1 = 0 \\ x + y + 2 = 0 \end{cases} \implies (x_0, y_0) = \left(-\frac{11}{8}, -\frac{5}{8}\right)$.
   - Translate: $X = x + \frac{11}{8}$, $Y = y + \frac{5}{8} \implies \frac{dY}{dX} = \frac{-3X+5Y}{X+Y}$, which is homogeneous.
   - Substitute $Y = vX$: $v + X\frac{dv}{dX} = \frac{-3+5v}{1+v} \implies X\frac{dv}{dX} = \frac{-3+5v}{1+v} - v = -\frac{(v-1)(v-3)}{1+v}$.
   - Separate variables: $\frac{1+v}{(v-1)(v-3)}\, dv = -\frac{dX}{X}$. Partial fractions: $\frac{1+v}{(v-1)(v-3)} = -\frac{1}{v-1} + \frac{2}{v-3}$.
   - Integrate: $\int \left(-\frac{1}{v-1} + \frac{2}{v-3}\right) dv = -\int \frac{dX}{X} \implies -\ln|v-1| + 2\ln|v-3| = -\ln|X| + C \implies$ $\ln\left(\frac{(v-3)^2}{|v-1|}\right) = -\ln|X| + C \implies X\,\frac{(v-3)^2}{v-1} = C_1$ (absorbing sign).
   - Back-substitute $v = Y/X$: $X\,\frac{(Y/X-3)^2}{Y/X-1} = \frac{(Y-3X)^2}{Y-X} = C_1$.
   - Return to x, y: $X = x + \frac{11}{8}$, $Y = y + \frac{5}{8} \implies \frac{(y-3x-\frac{7}{2})^2}{y-x-\frac{3}{4}} = C_1$.
   - Multiply numerator and denominator to clear fractions: $(2y - 6x - 7)^2 = C\,(4y - 4x - 3)$, where C is an arbitrary constant.
   - Final implicit solution: $\boxed{(2y-6x-7)^2 = C\,(4y-4x-3)}$.

15. $(x + 3y)\, dx - (3x + y - 6)\, dy = 0$

   - Rewrite as $\frac{dy}{dx} = \frac{x+3y}{3x+y-6}$. Here $a = 1$, $b = 3$, $c = 0$, $d = 3$, $e = 1$, $f = -6$. Compute $ae - bd = 1 \cdot 1 - 3 \cdot 3 = -8 \neq 0 \implies$ intersecting lines.
   - Find intersection: solve $\begin{cases} x + 3y = 0 \\ 3x + y - 6 = 0 \end{cases} \implies (x_0, y_0) = \left(\frac{9}{4}, -\frac{3}{4}\right)$.
   - Translate: $X = x - \frac{9}{4}$, $Y = y + \frac{3}{4} \implies \frac{dY}{dX} = \frac{X+3Y}{3X+Y}$, which is homogeneous.

- Substitute $Y = vX$: $v + X\frac{dv}{dX} = \frac{1+3v}{3+v} \implies X\frac{dv}{dX} = \frac{1+3v}{3+v} - v = \frac{1-v^2}{3+v}$.
- Separate variables: $\frac{3+v}{1-v^2}\,dv = \frac{dX}{X}$. Using partial fractions $\frac{3+v}{1-v^2} = \frac{2}{1-v} + \frac{1}{1+v}$.
- Integrate: $\int \frac{2}{1-v}\,dv + \int \frac{1}{1+v}\,dv = \int \frac{dX}{X} \implies -2\ln|1-v| + \ln|1+v| = \ln|X| + C \implies \ln\left|\frac{1+v}{(1-v)^2}\right| = \ln|X| + C \implies \frac{1+v}{(1-v)^2} = KX$.
- Back-substitute $v = Y/X$: $\frac{1+Y/X}{(1-Y/X)^2} = KX \implies$ multiply numerator and denominator by X^2: $\frac{X+Y}{(X-Y)^2} = K$ (after canceling X assuming $X \neq 0$).
- Return to x, y: $X = x - \frac{9}{4}$, $Y = y + \frac{3}{4} \implies X + Y = x + y - \frac{3}{2}$, $X - Y = x - y - 3$. Thus $\frac{x+y-\frac{3}{2}}{(x-y-3)^2} = K$.
- Multiply both sides by 2 to clear fraction: $2x + 2y - 3 = 2K(x-y-3)^2$.
- Final implicit solution: $\boxed{2x + 2y - 3 = C\,(x-y-3)^2}$, where C is an arbitrary constant.

16. $(x - y - 1)\,dx + (x + y + 3)\,dy = 0$

   - $\frac{dy}{dx} = -\frac{x-y-1}{x+y+3}$. $a = -1$, $b = 1$, $c = 1$, $d = 1$, $e = 1$, $f = 3$. $ae - bd = (-1)\cdot 1 - 1 \cdot 1 = -2 \neq 0 \implies$ intersecting.
   - Intersection: $\begin{cases} x - y - 1 = 0 \\ x + y + 3 = 0 \end{cases} \implies (x_0, y_0) = (-1, -2)$.
   - Translate: $X = x + 1$, $Y = y + 2 \implies \frac{dY}{dX} = -\frac{X-Y}{X+Y}$, homogeneous.
   - $Y = vX$: $v + X\frac{dv}{dX} = -\frac{1-v}{1+v} \implies X\frac{dv}{dX} = -\frac{1-v}{1+v} - v = -\frac{1+v^2}{1+v}$.
   - Separate: $\frac{1+v}{1+v^2}\,dv = -\frac{dX}{X} \implies \left(\frac{1}{1+v^2} + \frac{v}{1+v^2}\right)dv = -\frac{dX}{X}$.
   - Integrate: $\arctan v + \frac{1}{2}\ln(1+v^2) = -\ln|X| + C$.
   - Multiply by 2: $2\arctan v + \ln(1+v^2) = -2\ln|X| + C_1 \implies \ln\left(X^2(1+v^2)\right) = -2\arctan v + C_1 \implies X^2(1+v^2) = Ke^{-2\arctan v}$.
   - Back-substitute $v = Y/X$: $X^2\left(1 + \frac{Y^2}{X^2}\right) = X^2 + Y^2 = Ke^{-2\arctan(Y/X)}$.
   - Return to x, y: $X = x + 1$, $Y = y + 2 \implies \boxed{(x+1)^2 + (y+2)^2 = Ce^{-2\arctan\frac{y+2}{x+1}}}$.

Exercise 10: Solutions to Bernoulli Differential Equations

1. $y' + xy = x^3y^3$

   - This is a Bernoulli equation with $n = 3$. Divide by y^3:
   $$y^{-3}y' + xy^{-2} = x^3.$$
   - Substitute $z = y^{-2}$; then $z' = -2y^{-3}y' \implies y^{-3}y' = -\frac{1}{2}z'$.
   $$-\frac{1}{2}z' + xz = x^3 \quad \implies \quad z' - 2xz = -2x^3.$$
   - Integrating factor: $\mu(x) = e^{\int -2x\,dx} = e^{-x^2}$.
   $$\frac{d}{dx}\left(e^{-x^2}z\right) = -2x^3e^{-x^2}.$$
   - Integrate the right-hand side using $u = x^2$, $du = 2x\,dx$:
   $$\int -2x^3e^{-x^2}\,dx = -\int ue^{-u}du = (u+1)e^{-u} + C = (x^2+1)e^{-x^2} + C.$$
   Hence $e^{-x^2}z = (x^2+1)e^{-x^2} + C$.

- Solve for z: $z = x^2 + 1 + Ce^{x^2}$.
- Back-substitute $z = y^{-2}$:

$$y^{-2} = x^2 + 1 + Ce^{x^2} \quad \Longrightarrow \quad y^2 = \frac{1}{x^2 + 1 + Ce^{x^2}}.$$

- The answer is $\boxed{y^2\left(x^2 + 1 + Ce^{x^2}\right) = 1}$.

2. $(1 - x^2)y' - xy - axy^2 = 0$

- Rewrite as $y' - \dfrac{x}{1 - x^2}\,y = \dfrac{ax}{1 - x^2}\,y^2$. This is Bernoulli with $n = 2$.
- Let $z = y^{-1}$; then $z' = -y^{-2}y'$ and $y^{-2}y' = -z'$. Multiply the equation by y^{-2}:

$$-z' - \frac{x}{1 - x^2}z = \frac{ax}{1 - x^2} \quad \Longrightarrow \quad z' + \frac{x}{1 - x^2}z = -\frac{ax}{1 - x^2}.$$

- Integrating factor: $\mu(x) = \exp\Big(\int \frac{x}{1-x^2}dx\Big) = \exp\big(-\frac{1}{2}\ln|1 - x^2|\big) = (1 - x^2)^{-1/2}$.

$$\frac{d}{dx}\big((1 - x^2)^{-1/2}z\big) = -\frac{ax}{(1 - x^2)^{3/2}}.$$

- Integrate the right-hand side using $u = 1 - x^2$, $du = -2x\,dx$:

$$-\int \frac{ax}{(1 - x^2)^{3/2}}dx = \frac{a}{2}\int u^{-3/2}du = -a\,u^{-1/2} + C = -\frac{a}{\sqrt{1 - x^2}} + C.$$

Hence $(1 - x^2)^{-1/2}z = -\frac{a}{\sqrt{1-x^2}} + C$.
- Solve for z: $z = -a + C\sqrt{1 - x^2}$.
- Back-substitute $z = y^{-1}$:

$$y^{-1} = C\sqrt{1 - x^2} - a \quad \Longrightarrow \quad (C\sqrt{1 - x^2} - a)\,y = 1.$$

- The answer is $\boxed{(C\sqrt{1 - x^2} - a)\,y = 1}$.

3. $y' + \dfrac{2}{x}\,y = x^2y^4$

- This is a Bernoulli equation with $n = 4$. Divide by y^4:

$$y^{-4}y' + \frac{2}{x}y^{-3} = x^2.$$

- Substitute $z = y^{-3}$; then $z' = -3y^{-4}y' \implies y^{-4}y' = -\frac{1}{3}z'$.

$$-\frac{1}{3}z' + \frac{2}{x}z = x^2 \quad \Longrightarrow \quad z' - \frac{6}{x}z = -3x^2.$$

- Integrating factor: $\mu(x) = e^{\int -\frac{6}{x}dx} = e^{-6\ln x} = x^{-6}$.

$$\frac{d}{dx}\left(x^{-6}z\right) = -3x^{-4}.$$

- Integrate: $x^{-6}z = \int -3x^{-4}dx = x^{-3} + C$. Hence $z = x^3 + Cx^6$.
- Back-substitute $z = y^{-3}$:

$$y^{-3} = x^3 + Cx^6 \quad \Longrightarrow \quad y^3 = \frac{1}{x^3(1 + Cx^3)}.$$

- The solution can be written as $\boxed{y^3 = \dfrac{1}{x^3(1 + Cx^3)}}$ or equivalently $y^{-3} = x^3(1 + Cx^3)$.

4. $xy' + y = x^2y^2$

- Rewrite as $y' + \frac{1}{x}y = xy^2$. This is Bernoulli with $n = 2$.
- Divide by y^2: $y^{-2}y' + \frac{1}{x}y^{-1} = x$.
- Let $z = y^{-1}$; then $z' = -y^{-2}y' \implies y^{-2}y' = -z'$.

$$-z' + \frac{1}{x}z = x \quad \implies \quad z' - \frac{1}{x}z = -x.$$

- Integrating factor: $\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = x^{-1}$.

$$\frac{d}{dx}\big(x^{-1}z\big) = -1.$$

- Integrate: $x^{-1}z = -x + C \implies z = -x^2 + Cx$.
- Back-substitute $z = y^{-1}$:

$$y^{-1} = Cx - x^2 \quad \implies \quad y = \frac{1}{Cx - x^2}.$$

- Answer: $\boxed{y = \frac{1}{Cx - x^2}}$.

5. $y' + y = e^xy^2$

- This is Bernoulli with $n = 2$. Divide by y^2: $y^{-2}y' + y^{-1} = e^x$.
- Let $z = y^{-1}$; then $z' = -y^{-2}y' \implies y^{-2}y' = -z'$.

$$-z' + z = e^x \quad \implies \quad z' - z = -e^x.$$

- Integrating factor: $\mu(x) = e^{\int -1\,dx} = e^{-x}$.

$$\frac{d}{dx}\big(e^{-x}z\big) = -1.$$

- Integrate: $e^{-x}z = -x + C \implies z = (C - x)e^x$.
- Back-substitute $z = y^{-1}$:

$$y^{-1} = (C - x)e^x \quad \implies \quad y = \frac{e^{-x}}{C - x}.$$

- Answer: $\boxed{y = \frac{e^{-x}}{C - x}}$.

6. $2xy\,y' + (1 + x)y^2 = e^x$

- This can be transformed by noting that $2xyy'$ is the derivative of xy^2. Let $u = y^2$; then $u' = 2yy'$. The equation becomes $xu' + (1 + x)u = e^x$ (linear in u).
- Rewrite as $u' + \frac{1+x}{x}u = \frac{e^x}{x}$.
- Integrating factor: $\mu(x) = e^{\int \frac{1+x}{x}dx} = e^{\int(\frac{1}{x}+1)dx} = e^{\ln x + x} = xe^x$.

$$\frac{d}{dx}\big(xe^xu\big) = e^x \cdot xe^x = xe^{2x}.$$

- Integrate: $xe^xu = \int xe^{2x}dx$. Using integration by parts, $\int xe^{2x}dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$. Hence $xe^xu = \frac{e^{2x}}{4}(2x - 1) + C$.
- Solve for u: $u = \frac{e^x}{4x}(2x - 1) + \frac{C}{x}e^{-x}$.

- Back-substitute $u = y^2$:

$$y^2 = \frac{e^x}{4x}(2x-1) + \frac{C}{x}e^{-x}.$$

- Answer: $\boxed{y^2 = \frac{e^x}{4x}(2x-1) + \frac{C}{x}e^{-x}}$.

7. $y' + y\tan x = y^4 \sec x$

- This is Bernoulli with $n = 4$. Divide by y^4: $y^{-4}y' + \tan x\, y^{-3} = \sec x$.
- Let $z = y^{-3}$; then $z' = -3y^{-4}y' \implies y^{-4}y' = -\frac{1}{3}z'$.

$$-\frac{1}{3}z' + \tan x\, z = \sec x \quad \implies \quad z' - 3\tan x\, z = -3\sec x.$$

- Integrating factor: $\mu(x) = e^{\int -3\tan x\, dx} = e^{3\ln|\cos x|} = \cos^3 x$.

$$\frac{d}{dx}\left(\cos^3 x\, z\right) = -3\cos^3 x \sec x = -3\cos^2 x.$$

- Integrate: $\cos^3 x\, z = -3\int \cos^2 x\, dx = -3\int \frac{1+\cos 2x}{2}dx = -\frac{3}{2}x - \frac{3}{4}\sin 2x + C$.
- Hence $z = \frac{1}{\cos^3 x}\left(C - \frac{3}{2}x - \frac{3}{4}\sin 2x\right)$.
- Back-substitute $z = y^{-3}$:

$$y^{-3} = \frac{1}{\cos^3 x}\left(C - \frac{3}{2}x - \frac{3}{4}\sin 2x\right) \quad \implies \quad y^3 = \frac{\cos^3 x}{C - \frac{3}{2}x - \frac{3}{4}\sin 2x}.$$

- Simplify using $\sin 2x = 2\sin x\cos x$. Answer: $\boxed{y^3 = \frac{\cos^3 x}{C - \frac{3}{2}x - \frac{3}{2}\sin x\cos x}}$.

8. $xy' + y = x^3y^6$

- This is Bernoulli with $n = 6$. Divide by y^6: $xy^{-6}y' + y^{-5} = x^3$.
- Let $z = y^{-5}$; then $z' = -5y^{-6}y' \implies y^{-6}y' = -\frac{1}{5}z'$.

$$-\frac{x}{5}z' + z = x^3 \quad \implies \quad z' - \frac{5}{x}z = -5x^2.$$

- Integrating factor: $\mu(x) = e^{\int -\frac{5}{x}dx} = e^{-5\ln x} = x^{-5}$.

$$\frac{d}{dx}\left(x^{-5}z\right) = -5x^{-3}.$$

- Integrate: $x^{-5}z = -5\int x^{-3}dx = \frac{5}{2}x^{-2} + C$. Hence $z = \frac{5}{2}x^3 + Cx^5$.
- Back-substitute $z = y^{-5}$:

$$y^{-5} = \frac{5}{2}x^3 + Cx^5 \quad \implies \quad y^5 = \frac{1}{\frac{5}{2}x^3 + Cx^5} = \frac{2}{x^3(5 + 2Cx^2)}.$$

- Answer: $\boxed{y^5 = \frac{2}{x^3(5 + C_1x^2)}}$ where $C_1 = 2C$.

9. $3y^2y' - ay^3 - x - 1 = 0$

- This can be written as $y' - \frac{a}{3}\,y = \frac{x+1}{3}\,y^{-2}$. It is Bernoulli with $n = -2$.
- Set $z = y^3$; then $z' = 3y^2y'$. The original equation becomes $z' - az = x + 1$ (linear).
- Integrating factor: $\mu(x) = e^{-ax}$.

$$\frac{d}{dx}\left(e^{-ax}z\right) = (x+1)e^{-ax}.$$

- Integrate the right-hand side by parts:

$$\int (x+1)e^{-ax}dx = -\frac{x+1}{a}e^{-ax} - \frac{1}{a^2}e^{-ax} + C.$$

So $e^{-ax}z = -\left(\frac{x+1}{a} + \frac{1}{a^2}\right)e^{-ax} + C$.

- Solve for z: $z = -\frac{x+1}{a} - \frac{1}{a^2} + Ce^{ax}$.
- Back-substitute $z = y^3$ and multiply by a^2:

$$a^2y^3 = -a(x+1) - 1 + Ca^2e^{ax}.$$

Renaming Ca^2 as a new constant gives the answer $\boxed{a^2y^3 = Ce^{ax} - a(x+1) - 1}$.

10. $y'(x^2y^3 + xy) = 1$

- Rewrite as $\dfrac{dx}{dy} = x^2y^3 + xy$, i.e. $\dfrac{dx}{dy} - xy = x^2y^3$. This is a Bernoulli equation in $x(y)$ with $n = 2$.
- Let $z = x^{-1}$; then $z' = -x^{-2}\dfrac{dx}{dy}$. Dividing the equation by x^2 gives

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \quad\Longrightarrow\quad -z' - yz = y^3 \quad\Longrightarrow\quad z' + yz = -y^3.$$

- Integrating factor: $\mu(y) = e^{\int y\,dy} = e^{y^2/2}$.

$$\frac{d}{dy}\left(e^{y^2/2}z\right) = -y^3e^{y^2/2}.$$

- Integrate the right-hand side using $u = y^2/2$, $du = y\,dy$, $y^3dy = 2u\,du$:

$$\int -y^3e^{y^2/2}dy = -\int 2ue^u du = -2(u-1)e^u + C = -(y^2-2)e^{y^2/2} + C.$$

Hence $e^{y^2/2}z = -(y^2-2)e^{y^2/2} + C$.

- Solve for z: $z = 2 - y^2 + Ce^{-y^2/2}$.
- Back-substitute $z = x^{-1}$ and rearrange:

$$\frac{1}{x} = 2 - y^2 + Ce^{-y^2/2} \quad\Longrightarrow\quad x\big[(2-y^2)e^{y^2/2} + C\big] = e^{y^2/2}.$$

- The answer is $\boxed{x\left[(2-y^2)e^{\frac{1}{2}y^2} + C\right] = e^{\frac{1}{2}y^2}}$.

11. $(y\operatorname{Log}x - 2)y\,dx = x\,dy$

- Write as $\dfrac{dy}{dx} = \dfrac{y}{x}(y\operatorname{Log}x - 2) = \dfrac{\operatorname{Log}x}{x}y^2 - \dfrac{2}{x}y$, i.e.

$$y' + \frac{2}{x}y = \frac{\operatorname{Log}x}{x}y^2.$$

This is Bernoulli with $n = 2$.

- Let $z = y^{-1}$; then $z' = -y^{-2}y'$. Multiply by y^{-2}:

$$-z' + \frac{2}{x}z = \frac{\operatorname{Log}x}{x} \quad\Longrightarrow\quad z' - \frac{2}{x}z = -\frac{\operatorname{Log}x}{x}.$$

- Integrating factor: $\mu(x) = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$.

$$\frac{d}{dx}\left(x^{-2}z\right) = -\frac{\operatorname{Log}x}{x^3}.$$

- Integrate: $\displaystyle\int -\frac{\ln x}{x^3}\,dx = \frac{2\ln x + 1}{4x^2} + C$ (integration by parts, details omitted). Hence $x^{-2}z = \frac{2\ln x+1}{4x^2} + C$.
- Solve for z: $z = \frac{2\ln x+1}{4} + Cx^2$.
- Back-substitute $z = y^{-1}$ and multiply by 4:

$$\frac{4}{y} = 2\ln x + 1 + 4Cx^2 \quad \Longrightarrow \quad y\big(4Cx^2 + 2\ln x + 1\big) = 4.$$

  Renaming $4C$ as C and using $\operatorname{Log} x^2 = 2\ln x$ yields the answer $\boxed{y\,(Cx^2 + \operatorname{Log} x^2 + 1) = 4}$.

12. $y - y'\cos x = y^2\cos x(1 - \sin x)$
    - Rearrange: $y'\cos x = y - y^2\cos x(1-\sin x) \Longrightarrow y' = \sec x\, y - (1-\sin x)y^2$. So $y' - \sec x\, y = -(1-\sin x)y^2$, a Bernoulli equation with $n = 2$.
    - Set $z = y^{-1}$; then $z' = -y^{-2}y'$. Multiply by y^{-2}:

$$-z' - \sec x\, z = -(1 - \sin x) \quad \Longrightarrow \quad z' + \sec x\, z = 1 - \sin x.$$

    - Integrating factor: $\mu(x) = e^{\int \sec x\, dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$.

$$\frac{d}{dx}\big((\sec x + \tan x)z\big) = (\sec x + \tan x)(1 - \sin x).$$

    - Simplify the right-hand side:

$$(\sec x + \tan x)(1 - \sin x) = \frac{1 + \sin x}{\cos x}(1 - \sin x) = \frac{1 - \sin^2 x}{\cos x} = \frac{\cos^2 x}{\cos x} = \cos x.$$

    - Integrate: $(\sec x + \tan x)z = \int \cos x\, dx = \sin x + C$.
    - Solve for z: $z = \dfrac{\sin x + C}{\sec x + \tan x}$.
    - Since $\sec x + \tan x = \dfrac{1 + \sin x}{\cos x}$, we have $z = \dfrac{(\sin x + C)\cos x}{1 + \sin x}$.
    - Back-substitute $z = y^{-1}$:

$$y = \frac{1 + \sin x}{(\sin x + C)\cos x} = \frac{\tan x + \sec x}{\sin x + C}.$$

    - The answer is $\boxed{y = \dfrac{\tan x + \sec x}{\sin x + C}}$.

Exercises 11: Second-Order ODEs Reducible to First Order

Solve the following differential equations using the appropriate substitution (missing dependent or independent variable).

1. $y'' + 2y' = 4x$ (missing y)
    - Since y is missing, let $p = y'$, then $y'' = p'$. The equation becomes

$$p' + 2p = 4x,$$

      a linear first-order equation.
    - Integrating factor: $\mu(x) = e^{\int 2\, dx} = e^{2x}$.
    - Multiply the equation by μ:

$$\frac{d}{dx}\big(pe^{2x}\big) = 4xe^{2x}.$$

- Integrate both sides:

$$pe^{2x} = 4\int xe^{2x}\,dx + C_1 = 4\left(\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right) + C_1 = (2x-1)e^{2x} + C_1.$$

- Thus

$$p = \frac{dy}{dx} = 2x - 1 + C_1 e^{-2x}.$$

- Integrate again:

$$y = \int (2x-1)\,dx + C_1\int e^{-2x}\,dx = x^2 - x - \frac{C_1}{2}e^{-2x} + C_2.$$

- Renaming constants, let $A = -\frac{C_1}{2}$ and $B = C_2$. The general solution is

$$\boxed{y = x^2 - x + Ae^{-2x} + B}.$$

2. $1 + yy'' + (y')^2 = 0$ (missing x)

- Since x is missing, let $y' = p$ and use $y'' = p\frac{dp}{dy}$. Substituting,

$$1 + yp\frac{dp}{dy} + p^2 = 0.$$

- Rearrange:

$$yp\frac{dp}{dy} = -(1+p^2).$$

- Separate variables:

$$\frac{p}{1+p^2}\,dp = -\frac{dy}{y}.$$

- Integrate:

$$\frac{1}{2}\ln(1+p^2) = -\ln|y| + \frac{1}{2}\ln a^2,$$

where $\frac{1}{2}\ln a^2$ is an arbitrary constant. This simplifies to

$$\ln\big((1+p^2)y^2\big) = \ln a^2 \quad\Longrightarrow\quad (1+p^2)y^2 = a^2.$$

- Hence

$$p = \frac{dy}{dx} = \pm\frac{\sqrt{a^2-y^2}}{y}.$$

- Separate variables:

$$\pm\frac{y}{\sqrt{a^2-y^2}}\,dy = dx.$$

- Integrate:

$$\mp\sqrt{a^2-y^2} = x + b,$$

where b is another constant.

- Squaring gives the general solution:

$$\boxed{(x+b)^2 + y^2 = a^2}.$$

3. $y'' = 1 + (y')^2$ (missing x)

- Since x is missing, set $p = y'$ and use $y'' = p\frac{dp}{dy}$. Substituting gives

$$p\frac{dp}{dy} = 1 + p^2.$$

- Separate variables:

$$\frac{p}{1+p^2}\,dp = dy.$$

- Integrate:

$$\frac{1}{2}\ln(1+p^2) = y + C_1.$$

- Exponentiate:

$$1+p^2 = e^{2(y+C_1)} = C_2e^{2y}, \quad C_2 = e^{2C_1}.$$

- Then

$$p = \frac{dy}{dx} = \pm\sqrt{C_2e^{2y}-1}.$$

- Separate variables:

$$\frac{dy}{\pm\sqrt{C_2e^{2y}-1}} = dx.$$

- This integral can be evaluated by substitution $u = e^{-y}$ or recognized as leading to an inverse hyperbolic function. Alternatively, set $u = e^{-y}$, then $du = -e^{-y}dy = -udy$, so $dy = -du/u$. Then

$$\int \frac{dy}{\sqrt{C_2e^{2y}-1}} = \int \frac{-du/u}{\sqrt{C_2u^{-2}-1}} = -\int \frac{du}{\sqrt{C_2-u^2}}.$$

- Hence

$$\pm\left(-\int \frac{du}{\sqrt{C_2-u^2}}\right) = x + C_3 \implies \mp\arcsin\frac{u}{\sqrt{C_2}} = x + C_3.$$

- Returning to y, $u = e^{-y}$, so

$$\arcsin\left(\frac{e^{-y}}{\sqrt{C_2}}\right) = \mp(x+C_3).$$

- Taking sine of both sides and simplifying yields an explicit solution. A simpler final form can be obtained by choosing constants appropriately. The general solution is often written as

$$y = \ln\cosh(x+C) + D.$$

Verify: if $y = \ln\cosh(x+C)+D$, then $y' = \tanh(x+C)$, $y'' = \text{sech}^2(x+C) = 1-\tanh^2(x+C) = 1-(y')^2$, which satisfies the equation. So the solution is

$$\boxed{y = \ln\cosh(x+C_1) + C_2}.$$

4. $xy'' + y' = 0$ (missing y)

- Since y is missing, set $p = y'$, then $y'' = p'$. The equation becomes

$$xp' + p = 0.$$

- This is a first-order linear equation in p. It can be written as

$$\frac{d}{dx}(xp) = 0 \quad (\text{since } (xp)' = xp' + p).$$

- Integrate:

$$xp = C_1 \implies p = \frac{C_1}{x}.$$

- Then

$$\frac{dy}{dx} = \frac{C_1}{x} \implies y = C_1\ln|x| + C_2.$$

- The general solution is

$$\boxed{y = C_1\ln|x| + C_2}.$$

5. $y'' + 2y(y')^3 = 0$ (missing x)

- Since x is missing, set $p = y'$, then $y'' = p\frac{dp}{dy}$. Substituting,

$$p\frac{dp}{dy} + 2yp^3 = 0.$$

- If $p = 0$, then $y = \text{constant}$ is a solution. Otherwise, divide by p (assuming $p \neq 0$):

$$\frac{dp}{dy} + 2yp^2 = 0.$$

- This is a Bernoulli equation in p with $n = 2$. Rewrite as

$$\frac{dp}{dy} = -2yp^2.$$

- Separate variables:

$$\frac{dp}{p^2} = -2y\,dy.$$

- Integrate:

$$-\frac{1}{p} = -y^2 + C_1 \quad \Longrightarrow \quad \frac{1}{p} = y^2 - C_1.$$

- Hence

$$p = \frac{dy}{dx} = \frac{1}{y^2 - C_1}.$$

- Separate variables:

$$(y^2 - C_1)\,dy = dx.$$

- Integrate:

$$\frac{y^3}{3} - C_1 y = x + C_2.$$

- The general solution (including the constant solution $y = \text{constant}$) is

$$\boxed{\frac{y^3}{3} - C_1 y = x + C_2}.$$

6. $(1 + x^2)y'' + 2xy' = 0$ (missing y)
   - Since y is missing, set $p = y'$, then $y'' = p'$. The equation becomes

   $$(1 + x^2)p' + 2xp = 0.$$

   - This is a first-order linear (or separable) equation. Write as

   $$\frac{d}{dx}\big((1 + x^2)p\big) = 0 \quad (\text{since } ((1 + x^2)p)' = (1 + x^2)p' + 2xp).$$

   - Integrate:

   $$(1 + x^2)p = C_1 \quad \Longrightarrow \quad p = \frac{C_1}{1 + x^2}.$$

   - Then

   $$\frac{dy}{dx} = \frac{C_1}{1 + x^2} \quad \Longrightarrow \quad y = C_1 \arctan x + C_2.$$

   - The general solution is

   $$\boxed{y = C_1 \arctan x + C_2}.$$

7. $y'' = (y')^2$ (missing x and y? Actually both missing, but we can use either method)
   - This equation does not contain x or y explicitly. We can set $p = y'$, then $y'' = p'$. The equation becomes

   $$p' = p^2.$$

- Separate variables:

$$\frac{dp}{p^2} = dx.$$

- Integrate:

$$-\frac{1}{p} = x + C_1 \quad \Longrightarrow \quad p = -\frac{1}{x + C_1}.$$

- Then

$$\frac{dy}{dx} = -\frac{1}{x + C_1} \quad \Longrightarrow \quad y = -\ln|x + C_1| + C_2.$$

- The general solution is

$$\boxed{y = C_2 - \ln|x + C_1|}.$$

8. $yy'' = (y')^2$ (missing x)

- Since x is missing, set $p = y'$, then $y'' = p\frac{dp}{dy}$. Substituting,

$$yp\frac{dp}{dy} = p^2.$$

- If $p = 0$, then $y =$ constant is a solution. Otherwise, divide by p:

$$y\frac{dp}{dy} = p.$$

- This is a separable equation:

$$\frac{dp}{p} = \frac{dy}{y}.$$

- Integrate:

$$\ln|p| = \ln|y| + C_1 \quad \Longrightarrow \quad p = C_1 y, \quad C_1 \neq 0.$$

- Then

$$\frac{dy}{dx} = C_1 y \quad \Longrightarrow \quad \frac{dy}{y} = C_1 dx.$$

- Integrate:

$$\ln|y| = C_1 x + C_2 \quad \Longrightarrow \quad y = C_2 e^{C_1 x}.$$

- The constant solution $y = 0$ can be included by allowing $C_2 = 0$. Thus the general solution is

$$\boxed{y = C_2 e^{C_1 x}} \quad \text{or equivalently } y = Ae^{Bx}.$$

Exercises 12: Characteristic Equations of Linear Homogeneous ODEs

Take: $y = e^{rx}$,

1. $y'' - 3y' + 2y = 0 \implies r^2 - 3r + 2 = 0$
2. $y'' + 4y' + 4y = 0 \implies r^2 + 4r + 4 = 0$
3. $y'' + 2y' + 5y = 0 \implies r^2 + 2r + 5 = 0$
4. $y''' - 6y'' + 11y' - 6y = 0 \implies r^3 - 6r^2 + 11r - 6 = 0$
5. $y^{(4)} - y = 0 \implies r^4 - 1 = 0$
6. $y''' + 3y'' + 3y' + y = 0 \implies r^3 + 3r^2 + 3r + 1 = 0$
7. $y'' + 9y = 0 \implies r^2 + 9 = 0$
8. $y'' - 2y' + y = 0 \implies r^2 - 2r + 1 = 0$
9. $y''' - y' = 0 \implies r^3 - r = 0$
10. $y^{(4)} + 2y'' + y = 0 \implies r^4 + 2r^2 + 1 = 0$
11. $y''' - 3y' + 2y = 0 \implies r^3 - 3r + 2 = 0$
12. $y^{(5)} - y^{(4)} = 0 \implies r^5 - r^4 = 0$
13. $y''' + 4y'' + 5y' = 0 \implies r^3 + 4r^2 + 5r = 0$
14. $y^{(4)} + 8y'' + 16y = 0 \implies r^4 + 8r^2 + 16 = 0$
15. $y'' + 2y' + 10y = 0 \implies r^2 + 2r + 10 = 0$

Exercises 13: Wronskian

1. $W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x \cdot 2e^{2x} - e^{2x} \cdot e^x = e^{3x} \neq 0$, so the functions are linearly independent.

2. $W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$, independent.

3. $W(x, x^2, x^3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = x \cdot \begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - x^2 \cdot \begin{vmatrix} 1 & 3x^2 \\ 0 & 6x \end{vmatrix} + x^3 \cdot \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix}$. Compute the minors: $\begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} = (2x)(6x) - (3x^2)(2) = 12x^2 - 6x^2 = 6x^2$. $\begin{vmatrix} 1 & 3x^2 \\ 0 & 6x \end{vmatrix} = 1 \cdot 6x - 0 = 6x$. $\begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} = 1 \cdot 2 - 0 = 2$. Hence $W = x \cdot 6x^2 - x^2 \cdot 6x + x^3 \cdot 2 = 6x^3 - 6x^3 + 2x^3 = 2x^3 \neq 0$ for $x \neq 0$. Since the functions are analytic, they are independent on any interval.

4. $W(e^{2x}, e^{-3x}) = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} = e^{2x}(-3e^{-3x}) - e^{-3x}(2e^{2x}) = -3e^{-x} - 2e^{-x} = -5e^{-x} \neq 0$.

5. $W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} = 2 \neq 0$.

6. $W(e^x, e^x \cos x, e^x \sin x)$. Factor e^x from each row:

$$W = e^{3x} \begin{vmatrix} 1 & \cos x & \sin x \\ 1 & \cos x - \sin x & \sin x + \cos x \\ 1 & -2\sin x & 2\cos x \end{vmatrix}$$

(since derivatives: $(e^x)' = e^x$, $(e^x \cos x)' = e^x(\cos x - \sin x)$, $(e^x \sin x)' = e^x(\sin x + \cos x)$, and second derivatives: $(e^x)'' = e^x$, $(e^x \cos x)'' = e^x(-2\sin x)$, $(e^x \sin x)'' = e^x(2\cos x)$.) Compute the determinant:

$$\Delta = \begin{vmatrix} 1 & \cos x & \sin x \\ 1 & \cos x - \sin x & \sin x + \cos x \\ 1 & -2\sin x & 2\cos x \end{vmatrix}$$

Subtract the first row from the second and third:

$$\begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -2\sin x - \cos x & 2\cos x - \sin x \end{vmatrix}$$

Then expand:

$$= 1 \cdot \begin{vmatrix} -\sin x & \cos x \\ -2\sin x - \cos x & 2\cos x - \sin x \end{vmatrix} = (-\sin x)(2\cos x - \sin x) - \cos x(-2\sin x - \cos x)$$

$$= -2\sin x \cos x + \sin^2 x + 2\sin x \cos x + \cos^2 x = \sin^2 x + \cos^2 x = 1$$

Therefore $W = e^{3x} \cdot 1 = e^{3x} \neq 0$, independent.

7. $W(\cosh x, \sinh x) = \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} = \cosh^2 x - \sinh^2 x = 1 \neq 0$.

8. $\ln x^2 = 2 \ln x$, so $f_2(x) = 2f_1(x)$, hence they are linearly dependent. $W = 0$.

9. $W(e^{ax}, e^{bx}) = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix} = e^{(a+b)x}(b - a) \neq 0$ if $a \neq b$.

10. x and $|x|$ on $\mathbb{R}$. The functions are not differentiable at $x = 0$, but we can compute the Wronskian on intervals $x > 0$ and $x < 0$. For $x > 0$, $|x| = x$, so $W = 0$. For $x < 0$, $|x| = -x$, so $f_2 = -f_1$, also $W = 0$. Thus the functions are linearly dependent on any interval (linear relation: $f_2 = \pm f_1$).

11. $f_2(x) = e^{x+1} = e \cdot e^x$, so it is a constant multiple of f_1, hence $W = 0$.

12. $\sin^2 x + \cos^2 x = 1$, so there is a linear relation: $f_1 + f_2 - f_3 = 0$, hence $W = 0$.

13. $W(1, x, e^x) = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} = e^x \neq 0.$

14. $W(x, xe^x) = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = x(e^x + xe^x) - xe^x \cdot 1 = xe^x + x^2e^x - xe^x = x^2e^x \neq 0$ for $x \neq 0$.

15. $W(\tan x, \sec x) = \begin{vmatrix} \tan x & \sec x \\ \sec^2 x & \sec x \tan x \end{vmatrix} = \tan x(\sec x \tan x) - \sec x(\sec^2 x) = \sec x \tan^2 x - \sec^3 x =$ $\sec x(\tan^2 x - \sec^2 x) = \sec x((\sec^2 x - 1) - \sec^2 x) = -\sec x \neq 0$ on the given interval.

Exercises 14: Reduction of Order

Given: $x^2y'' - xy' + y = 0,\ x > 0$, with known solution $y_1 = x$.

1. Write in standard form:
$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0, \quad p(x) = -\frac{1}{x}.$$

2. Use the reduction of order formula:
$$y_2 = y_1 \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx.$$

3. Compute:
$$\int p\,dx = \int -\frac{1}{x}\,dx = -\ln x, \quad e^{-\int p\,dx} = e^{\ln x} = x.$$
$$y_2 = x \int \frac{x}{x^2}\,dx = x \int \frac{1}{x}\,dx = x \ln x.$$

4. General solution:
$$y(x) = C_1 x + C_2\,x \ln x.$$

Exercise 15: Second-Order Linear Differential Equations

1. $y'' - 3y' + 2y = 0$

   Characteristic equation: $r^2 - 3r + 2 = 0 \Rightarrow r = 1, 2 \Rightarrow$ General solution: $\boxed{y = C_1e^x + C_2e^{2x}}$

2. $y'' + 5y' + 6y = 0$

   $r^2 + 5r + 6 = 0 \Rightarrow r = -2, -3 \Rightarrow \boxed{y = C_1e^{-2x} + C_2e^{-3x}}$

3. $y'' - 4y = 0$

   $r^2 - 4 = 0 \Rightarrow r = \pm 2 \Rightarrow \boxed{y = C_1e^{2x} + C_2e^{-2x}}$

4. $y'' - 6y' + 9y = 0$

   $r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 \Rightarrow \boxed{y = (C_1 + C_2x)e^{3x}}$

5. $y'' + 4y' + 4y = 0$

   $r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow \boxed{y = (C_1 + C_2x)e^{-2x}}$

6. $y'' + 4y = 0$

   $r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow \boxed{y = C_1 \cos 2x + C_2 \sin 2x}$

7. $y'' - 2y' + 5y = 0$

   $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i \Rightarrow \boxed{y = e^x(C_1 \cos 2x + C_2 \sin 2x)}$

8. $y'' + 2y' + 2y = 0$
$r^2 + 2r + 2 = 0 \Rightarrow r = -1 \pm i \Rightarrow \boxed{y = e^{-x}(C_1 \cos x + C_2 \sin x)}$

9. $y'' - y' - 2y = 4x^2$
Method 1 (Undetermined coefficients): $y_h = C_1 e^{2x} + C_2 e^{-x}$; $y_p = Ax^2 + Bx + C$ gives $A = -2,\ B = 2,\ C = -3$
$\boxed{y = C_1 e^{2x} + C_2 e^{-x} - 2x^2 + 2x - 3}$
Method 2 (Variation of parameters): $y_1 = e^{2x},\ y_2 = e^{-x}$; $W = y_1 y_2' - y_2 y_1' = -3e^x$
$u_1' = \frac{-y_2 f}{W} = \frac{4x^2}{3} e^{-2x},\ u_2' = \frac{y_1 f}{W} = -\frac{4x^2}{3} e^x$
Integrate: $u_1 = -\frac{2}{3} e^{-2x}\left(x^2 + x + \frac{1}{2}\right)$, $u_2 = -\frac{4}{3} e^x (x^2 - 2x + 2)$
Then $y_p = u_1 y_1 + u_2 y_2 = -2x^2 + 2x - 3$ (same as above).

10. $y'' + 3y' + 2y = 2x^2 + 1$
Method 1 (Undetermined coefficients): $y_h = C_1 e^{-x} + C_2 e^{-2x}$; $y_p = Ax^2 + Bx + C$ gives $A = 1,\ B = -3,\ C = 4$
$\boxed{y = C_1 e^{-x} + C_2 e^{-2x} + x^2 - 3x + 4}$
Method 2 (Variation of parameters): $y_1 = e^{-x},\ y_2 = e^{-2x}$; $W = -e^{-3x}$
$u_1' = \frac{-y_2 f}{W} = e^x (2x^2 + 1),\ u_2' = \frac{y_1 f}{W} = -e^{2x}(2x^2 + 1)$
Integrate: $u_1 = e^x (2x^2 - 4x + 5)$, $u_2 = -e^{2x}(x^2 - x + 1)$
Then $y_p = u_1 y_1 + u_2 y_2 = x^2 - 3x + 4$ (same as above).

11. $y'' - 5y' + 6y = e^{2x}$
Method 1: $y_h = C_1 e^{2x} + C_2 e^{3x}$; resonance $\Rightarrow y_p = Axe^{2x}$ gives $A = -1$
$\boxed{y = C_1 e^{2x} + C_2 e^{3x} - xe^{2x}}$
Method 2: $y_1 = e^{2x},\ y_2 = e^{3x}$; $W = e^{5x}$
$u_1' = -1,\ u_2' = e^{-x}$; integrate $u_1 = -x,\ u_2 = -e^{-x}$
Then $y_p = (-x)e^{2x} + (-e^{-x})e^{3x} = -xe^{2x} - e^{2x}$. Absorb $-e^{2x}$ into y_h to get $y_p = -xe^{2x}$.

12. $y'' + 2y' + y = e^{-x}$
Method 1: $y_h = (C_1 + C_2 x)e^{-x}$; double resonance $\Rightarrow y_p = Ax^2 e^{-x}$ gives $A = \frac{1}{2}$
$\boxed{y = (C_1 + C_2 x + \frac{1}{2}x^2)e^{-x}}$
Method 2: $y_1 = e^{-x},\ y_2 = xe^{-x}$; $W = e^{-2x}$
$u_1' = -x,\ u_2' = 1$; integrate $u_1 = -x^2/2,\ u_2 = x$
Then $y_p = (-\frac{x^2}{2})e^{-x} + x \cdot xe^{-x} = \frac{x^2}{2} e^{-x}$ (matches).

13. $y'' - 2y' + y = e^x + e^{-x}$
Method 1: $y_h = (C_1 + C_2 x)e^x$; for e^x: $y_{p1} = Ax^2 e^x$ with $A = \frac{1}{2}$; for e^{-x}: $y_{p2} = Be^{-x}$ with $B = \frac{1}{4}$
$\boxed{y = (C_1 + C_2 x + \frac{1}{2}x^2)e^x + \frac{1}{4}e^{-x}}$
Method 2: $y_1 = e^x,\ y_2 = xe^x$; $W = e^{2x}$
$u_1' = -x - xe^{-2x},\ u_2' = 1 + e^{-2x}$
Integrate: $u_1 = -\frac{x^2}{2} + \frac{x}{2}e^{-2x} + \frac{1}{4}e^{-2x}$, $u_2 = x - \frac{1}{2}e^{-2x}$
Then $y_p = u_1 e^x + u_2 xe^x = \frac{x^2}{2} e^x + \frac{1}{4} e^{-x}$ (matches).

14. $y'' + y = \sin x$
Method 1: $y_h = C_1 \cos x + C_2 \sin x$; resonance $\Rightarrow y_p = x(A\cos x + B \sin x)$ gives $A = -\frac{1}{2},\ B = 0$
$\boxed{y = C_1 \cos x + C_2 \sin x - \frac{x}{2} \cos x}$
Method 2: $y_1 = \cos x,\ y_2 = \sin x$; $W = 1$
$u_1' = -\sin^2 x,\ u_2' = \cos x \sin x$
Integrate: $u_1 = -\frac{x}{2} + \frac{1}{2} \sin x \cos x$, $u_2 = \frac{1}{2} \sin^2 x$
Then $y_p = (-\frac{x}{2} + \frac{1}{2} \sin x \cos x) \cos x + \frac{1}{2} \sin^2 x \sin x = -\frac{x}{2} \cos x + \frac{1}{2} \sin x (\cos^2 x + \sin^2 x) = -\frac{x}{2} \cos x + \frac{1}{2} \sin x$. Absorb $\frac{1}{2} \sin x$ into y_h to get $y_p = -\frac{x}{2} \cos x$.

15. $y'' + 4y = \cos 2x$

Method 1: $y_h = C_1 \cos 2x + C_2 \sin 2x$; resonance $\Rightarrow y_p = x(A\cos 2x + B\sin 2x)$ gives $A = 0,\ B = \frac{1}{4}$

$$\boxed{y = C_1 \cos 2x + C_2 \sin 2x + \frac{x}{4}\sin 2x}$$

Method 2: $y_1 = \cos 2x,\ y_2 = \sin 2x;\ W = 2$
$u_1' = -\frac{1}{2}\sin 2x \cos 2x = -\frac{1}{4}\sin 4x,\ u_2' = \frac{1}{2}\cos^2 2x = \frac{1}{4}(1 + \cos 4x)$
Integrate: $u_1 = \frac{1}{16}\cos 4x,\ u_2 = \frac{x}{4} + \frac{1}{16}\sin 4x$
Then $y_p = \frac{1}{16}\cos 4x \cos 2x + \left(\frac{x}{4} + \frac{1}{16}\sin 4x\right)\sin 2x = \frac{x}{4}\sin 2x + \frac{1}{16}\cos 2x$ (after simplification). Absorb $\frac{1}{16}\cos 2x$ into y_h to get $y_p = \frac{x}{4}\sin 2x$.

16. $y'' - 2y' + 2y = e^x \cos x$

Method 1: $y_h = e^x(C_1 \cos x + C_2 \sin x)$; resonance $\Rightarrow y_p = xe^x(A\cos x + B\sin x)$ gives $A = 0,\ B = \frac{1}{2}$

$$\boxed{y = e^x(C_1 \cos x + C_2 \sin x) + \frac{x}{2}e^x \sin x}$$

Method 2: $y_1 = e^x \cos x,\ y_2 = e^x \sin x;\ W = e^{2x}$
$u_1' = -\sin x \cos x = -\frac{1}{2}\sin 2x,\ u_2' = \cos^2 x = \frac{1}{2}(1 + \cos 2x)$
Integrate: $u_1 = \frac{1}{4}\cos 2x,\ u_2 = \frac{x}{2} + \frac{1}{4}\sin 2x$
Then $y_p = \frac{1}{4}\cos 2x e^x \cos x + \left(\frac{x}{2} + \frac{1}{4}\sin 2x\right) e^x \sin x = \frac{x}{2}e^x \sin x + \frac{1}{4}e^x \cos x$ (after simplification). Absorb $\frac{1}{4}e^x \cos x$ into y_h to get $y_p = \frac{x}{2}e^x \sin x$.

17. $y'' + 2y' + 2y = e^{-x} \sin x$

Method 1: $y_h = e^{-x}(C_1 \cos x + C_2 \sin x)$; resonance $\Rightarrow y_p = xe^{-x}(A\cos x + B\sin x)$ gives $A = -\frac{1}{2},\ B = 0$

$$\boxed{y = e^{-x}(C_1 \cos x + C_2 \sin x) - \frac{x}{2}e^{-x} \cos x}$$

Method 2: $y_1 = e^{-x} \cos x,\ y_2 = e^{-x} \sin x;\ W = e^{-2x}$
$u_1' = -\sin^2 x,\ u_2' = \cos x \sin x$
Integrate: $u_1 = -\frac{x}{2} + \frac{1}{2}\sin x \cos x,\ u_2 = \frac{1}{2}\sin^2 x$
Then $y_p = (-\frac{x}{2} + \frac{1}{2}\sin x \cos x)e^{-x}\cos x + \frac{1}{2}\sin^2 x e^{-x} \sin x = -\frac{x}{2}e^{-x}\cos x + \frac{1}{2}e^{-x}\sin x$ (after simplification). Absorb $\frac{1}{2}e^{-x}\sin x$ into y_h to get $y_p = -\frac{x}{2}e^{-x}\cos x$.

18. $y'' + y = x \cos x$

Method 1: $y_h = C_1 \cos x + C_2 \sin x$; resonance $\Rightarrow y_p = (Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x$ yields $A = 0,\ B = \frac{1}{4},\ C = \frac{1}{4},\ D = 0$

$$\boxed{y = C_1 \cos x + C_2 \sin x + \frac{x}{4}\cos x + \frac{x^2}{4}\sin x}$$

Method 2: $y_1 = \cos x,\ y_2 = \sin x;\ W = 1$
$u_1' = -x \sin x \cos x = -\frac{x}{2}\sin 2x,\ u_2' = x\cos^2 x = \frac{x}{2}(1 + \cos 2x)$
Integrate: $u_1 = \frac{x}{4}\cos 2x - \frac{1}{8}\sin 2x,\ u_2 = \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x$
Then $y_p = u_1 \cos x + u_2 \sin x = \frac{x}{4}\cos x + \frac{x^2}{4}\sin x - \frac{1}{8}\sin x$ (after simplification). Absorb $-\frac{1}{8}\sin x$ into y_h to get $y_p = \frac{x}{4}\cos x + \frac{x^2}{4}\sin x$.

19. $y'' + 4y = 12x, \quad y(0) = 2,\ y'(0) = 3$

$y_h = C_1 \cos 2x + C_2 \sin 2x$; $y_p = 3x$; general $y = C_1 \cos 2x + C_2 \sin 2x + 3x$
Initial conditions: $C_1 = 2,\ C_2 = 0$

$$\boxed{y = 2\cos 2x + 3x}$$

20. $y'' - y = 2e^x, \quad y(0) = 0,\ y'(0) = 1$

$y_h = C_1 e^x + C_2 e^{-x}$; resonance $\Rightarrow y_p = xe^x$; general $y = C_1 e^x + C_2 e^{-x} + xe^x$
Initial conditions: $C_1 = 0,\ C_2 = 0$

$$\boxed{y = xe^x}$$

21. $y'' + 2y' + y = 0, \quad y(0) = 1,\ y'(0) = -1$

$y = (C_1 + C_2 x)e^{-x}$; $y(0) = C_1 = 1$; $y'(0) = C_2 - C_1 = -1 \Rightarrow C_2 = 0$

$$\boxed{y = e^{-x}}$$

22. $y'' - 2y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 3$

$y = e^x(C_1 \cos 2x + C_2 \sin 2x)$; $y(0) = C_1 = 1$; $y'(0) = C_1 + 2C_2 = 3 \Rightarrow C_2 = 1$

$\boxed{y = e^x(\cos 2x + \sin 2x)}$