Solution of series N°2.

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Exercise 01

Using the Riemann sums, calculate the following integrals.

① $\int_0^1 x^2\,dx$.

we have:

$$\int_0^1 x^2\,dx = \lim_{n \to +\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)$$

$$= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2}$$

$$= \lim_{n \to +\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \lim_{n \to +\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}.$$

② $\int_0^1 e^x\,dx$

we have:

$$\int_0^1 e^x\,dx = \lim_{n \to +\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)$$

$$= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \left(e^{\frac{1}{n}}\right)^k \text{ (sum of the geometric sequence)}$$

$$= \lim_{n \to +\infty} \frac{1}{n} \left(e^{\frac{1}{n}} \frac{e-1}{e^{\frac{1}{n}}-1}\right) = e-1.$$

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③ $\int_1^2 x\, dx$.

$$\int_1^2 x\, dx = \lim_{n \to +\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)$$

$$= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \left(1 + \frac{k}{n}\right)$$

$$= \lim_{n \to +\infty} \left(\frac{1}{n} \sum_{k=1}^{n} 1\right) + \left(\frac{1}{n^2} \sum_{k=1}^{n} k\right) = \lim_{n \to +\infty} \left(1 + \frac{n+1}{2n}\right) = \frac{3}{2}$$

Exercise 02.

Using the Riemann sum determine the limit of the following sequences:

① $U_n = \sum_{k=0}^{n-1} \frac{n}{k^2 + n^2}$.

$$U_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + k^2} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{n^2}{n^2\left(1 + \frac{k^2}{n^2}\right)} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{1 + \left(\frac{k}{n}\right)^2}$$

$$= \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + k\,\frac{b-a}{n}\right)$$

where

$$a = 0, \quad b = 1, \quad f(x) = \frac{1}{1+x^2}.$$

So

$$\lim_{n \to +\infty} U_n = \int_0^1 \frac{1}{1+x^2}\, dx = \arctan x \Big]_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}$$

② $V_n = \sum_{k=0}^{n} \frac{n}{(n+k)^2}$

$$\lim_{n \to +\infty} V_n = \lim_{n \to +\infty} \sum_{k=0}^{n} \frac{n}{(n+k)^2} = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n} \frac{1}{\left(1 + \frac{k}{n}\right)^2} = \lim_{n \to +\infty} \frac{b-a}{n} \sum_{k=0}^{n} f\left(a + k\,\frac{b-a}{n}\right)$$

where $a = 1$, $b = 2$, $f(x) = \frac{1}{\ }$

So

$$\lim_{n\to+\infty} V_n = \int_1^2 \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right]_1^2 = \frac{1}{2}.$$

Exercise 03.

Calculate the following primitives:

① $\int x^2(1-x^3)^4\,dx$

$$\int x^2(1-x^3)^4\,dx = \frac{-1}{3}\int \underbrace{-3x^2}_{u'(x)}\,(\underbrace{1-x^3}_{u(x)})^4\,dx.$$

$$= \frac{-1}{3}\left[\frac{(1-x^3)^5}{5}\right] + C$$

$$= \frac{-1}{15}(1-x^3)^5 + C,\quad C\in\mathbb{R}.$$

② $\int \frac{\cos x}{(1+\sin x)^3}\,dx.$

$$\int \frac{\overbrace{\cos x}^{u'(x)}}{(\underbrace{1+\sin x}_{u(x)})^3}\,dx = \frac{-1}{2}\,\frac{1}{(1+\sin x)^2} + C,\quad C\in\mathbb{R}$$

③ $\int \frac{1}{x\ln x}\,dx.$

$$\int \frac{1}{x\ln x}\,dx = \int \frac{\overbrace{\frac{1}{x}}^{u'(x)}}{\underbrace{\ln x}_{u(x)}}\,dx = \ln|\ln x| + C,\quad C\in\mathbb{R}.$$

④ $\int x\, e^{-4x^2+9}\, dx$.

$$\int x\, e^{-4x^2+9}\, dx = \frac{-1}{8}\int \underbrace{-8x}_{u'(x)}\, e^{\overbrace{-4x^2+9}^{u(x)}}\, dx$$

$$= \frac{-1}{8}\, e^{-4x^2+9} + C \quad , \quad C \in \mathbb{R}.$$

⑤ $\int \frac{x^2}{\sqrt{1+2x^3}}\, dx$

$$\int \frac{x^2}{\sqrt{1+2x^3}}\, dx = \frac{1}{6}\int \frac{\overbrace{6x^2}^{u'(x)}}{\sqrt{\underbrace{1+2x^3}_{u(x)}}}\, dx = \frac{1}{6}\, 2\sqrt{1+2x^3} + C$$

$$= \frac{1}{3}\sqrt{1+2x^3} + C \quad , \quad C \in \mathbb{R}.$$

Exercise 04

By integration by parts, calculate the following integral

① $\int x^2 \sin x\, dx$.

we have:

$$\begin{cases} f(x) = x^2 \\ g'(x) = \sin x \end{cases} \Longrightarrow \begin{cases} f'(x) = 2x \\ g(x) = -\cos x. \end{cases}$$

So :

$$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx.$$

by parts :

$$\begin{cases} f(x) = 2x \\ g'(x) = \cos x \end{cases} \Longrightarrow \begin{array}{l} f'(x) = 2 \\ g(x) = \sin x. \end{array}$$

So

$$\int 2x \cos x = 2x \sin x - \int 2 \sin x \, dx = 2x \sin x + 2 \cos x + C.$$

Then

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C, \; C \in \mathbb{R}$$

$$= (2 - x^2) \cos x + 2x \sin x + C, \quad C \in \mathbb{R}.$$

② $\int (\ln x)^2 dx.$

we have :

$$\begin{cases} f(x) = \ln^2 x \\ g'(x) = 1 \end{cases} \Longrightarrow \begin{cases} f'(x) = 2 \frac{\ln x}{x}. \\ g(x) = x. \end{cases}$$

So

$$\int \ln^2 x \, dx = x \ln^2 x - \int 2 \ln x \, dx = x \ln^2 x - 2x \ln x + 2x + C, \; C \in \mathbb{R}.$$

③ $\displaystyle\int_0^1 x\, e^{2x}\, dx$.

we have :

$$\begin{cases} f(x) = x \\ g'(x) = e^{2x} \end{cases} \Longrightarrow \begin{cases} f'(x) = 1 \\ g(x) = \frac{1}{2} e^{2x}. \end{cases}$$

So

$$\int_0^1 x\, e^{2x}\, dx = \left[\frac{1}{2} x\, e^{2x}\right]_0^1 - \int_0^1 \frac{1}{2} e^{2x}\, dx.$$

$$= \frac{e^2}{2} - \left[\frac{1}{4} e^{2x}\right]_0^1 = \frac{e^2}{2} - \frac{1}{4} e^2 + \frac{1}{4}$$

$$= \frac{e^2 + 1}{4}.$$

④ $\displaystyle\int_0^1 x \arctan x\, dx$.

we have:

$$\begin{cases} f(x) = \arctan x \\ g'(x) = x \end{cases} \Longrightarrow \begin{cases} f'(x) = \frac{1}{1+x^2} \\ g(x) = \frac{x^2}{2} \end{cases}$$

So

$$\int_0^1 x \arctan x\, dx = \left.\frac{x^2}{2} \arctan x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\, dx.$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{1+x^2}\, dx = \frac{\pi}{8} - \frac{1}{2}\int_0^1 1\, dx + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\, dx = \frac{\pi}{8} - \left.\frac{x}{2}\right]_0^1 + \left.\frac{\arctan x}{2}\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}.$$

Exercise 5

① $\int \cos\sqrt{x}\, dx.$

we put $t = \sqrt{x} \Rightarrow x = t^2 \Rightarrow dx = 2t\,dt$

so $\int \cos\sqrt{x}\, dx = \int 2t \cos t\, dt.$

by parts :

$$\begin{cases} f(t) = 2t \\ g'(t) = \cos t \end{cases} \Longrightarrow \begin{cases} f'(t) = 2 \\ g(t) = \sin t \end{cases}$$

so.

$$\int \cos\sqrt{x}\, dx = \int 2t \cos t\, dt = 2t \sin t - \int 2\sin t\, dt.$$

$$= 2t \sin t + 2\cos t + C.$$

$$= 2\sqrt{x} \sin\sqrt{x} + 2\cos\sqrt{x} + C, \quad C \in \mathbb{R}.$$

② $\int \frac{x}{\sqrt{x+1}}\, dx.$

we put $t = \sqrt{x+1} \Rightarrow x = t^2 - 1 \Rightarrow dx = 2t\,dt$

so $\int \frac{x}{\sqrt{x+1}}\, dx = \int \frac{t^2-1}{t} \cdot 2t\, dt = \int 2t^2 - 2t\, dt = \frac{2}{3} t^3 - 2t + C$

$$= \frac{2}{3}\sqrt{x+1}^{\,3} - 2\sqrt{x+1} + C \quad , \quad C \in \mathbb{R}.$$

③ $\displaystyle\int_0^{\pi} \cos^4 x \sin x \, dx.$

we put $t = \cos x \Rightarrow dt = -\sin x \, dx \Rightarrow dx = \dfrac{-dt}{\sin x}$.

$$\begin{cases} \text{if } x = 0 \\ \text{if } x = \pi \end{cases} \Longrightarrow \begin{array}{l} t = \cos 0 = 1. \\ t = \cos \pi = -1. \end{array} \Longrightarrow \int_0^{\pi} dx = \int_1^{-1} dt = -\int_{-1}^{1} dt.$$

so

$$\int_0^{\pi} \cos^4 x \sin x \, dx = +\int_{-1}^{1} t^4 \, dt = \left[\frac{t^5}{5}\right]_{-1}^{1} = \frac{2}{5}.$$

④ $\displaystyle\int_1^{e} \frac{(\ln x)^n}{x} \, dx.$

we put $t = \ln x \Rightarrow x = e^t \Rightarrow dx = e^t \, dt$.

$$\begin{cases} \text{if } x = 1 \\ \text{if } x = e \end{cases} \Longrightarrow \begin{array}{l} t = 0 \\ t = 1 \end{array} \Longrightarrow \int_1^{e} dx = \int_0^{1} dt.$$

So

$$\int_1^{e} \frac{(\ln x)^n}{x} \, dx = \int_0^{1} \frac{t^n}{e^t} \, e^t \, dt = \int_0^{1} t^n \, dt = \left[\frac{t^{n+1}}{n+1}\right]_0^{1}$$

$$= \frac{1}{n+1}.$$