Solution of series N°1

Exercise 01.

① $\forall x \in [0, \frac{\pi}{2}]$, $x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} - \frac{x^5}{120}$

Using Taylor - Lagrange formula for $f(x) = \sin x$

$x_0 = 0$. and $n = 3$, $n = 5$ we get:

For $n=3$

$$\sin x = x - \frac{x^3}{6} + \frac{\sin c}{24} x^4 \text{ where } x \in [0, \tfrac{\pi}{2}] \text{ and } c \in]0, x[$$

For $n=5$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{\sin k}{720} x^6 \text{ where } x \in [0, \tfrac{\pi}{2}] \text{ and } k \in]0, x[$$

Since $x, c, k \in [0, \frac{\pi}{2}]$, $\frac{\sin c}{24} x^4 \geq 0$ and $-\frac{\sin k}{720} x^6 \leq 0$

we get: $\forall x \in [0, \frac{\pi}{2}]$ $\quad x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} - \frac{x^5}{120}$

② Exercise 02.

we show that: $\forall x > 0$, $x - \frac{x^2}{2} < \ln(1+x) < x$.

The Maclaurin's formula is

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1}$$

where $0 < \theta < 1$

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• we show that $\ln(1+x) > x - \frac{x^2}{2}$

we have: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(\theta x)}{3!}x^3, \; 0<\theta<1$

$$\begin{cases} f(x) = \ln(1+x) \implies f(0) = 0. \\ f'(x) = \frac{1}{1+x} \implies f'(0) = 1. \\ f''(x) = -\frac{1}{(1+x)^2} \implies f''(0) = -1. \\ f^{(3)}(x) = \frac{2}{(1+x)^3} \implies f^{(3)}(\theta x) = \frac{2}{(1+\theta x)^3} \quad 0<\theta<1. \end{cases}$$

So

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3(1+\theta x)^3}x^3.$$

since $x>0$ and $0<\theta<1$, $\frac{1}{3(1+\theta x)^3}x^3 > 0$

Then $x - \frac{x^2}{2} + \frac{1}{3(1+\theta x)^3}x^3 > x - \frac{x^2}{2}$

so $\ln(1+x) > x - \frac{x^2}{2}$.

• Now, we show that $\ln(1+x) < x$.

we have: $f(x) = f(0) + f'(0)x + \frac{f''(\theta x)}{2!}x^2, \; 0<\theta<1.$

$$\begin{cases} f(x) = \ln(1+x) \implies f(0) = 0 \\ f'(x) = \frac{1}{1+x} \implies f'(0) = 1 \end{cases}$$

$f''(x) = \frac{-1}{(1+x)^2} \implies f''(\theta x) = \frac{-1}{(1+\theta x)^2}$

So: $\ln(1+x) = x - \frac{x^2}{2(1+\theta x)^2}$, $0<\theta<1$.

since $x>0$ and $0<\theta<1$, $\frac{x^2}{2(1+\theta x)^2} > 0$.

Then $x - \frac{x^2}{2(1+\theta x)^2} < x$

so $\ln(1+x) < x$.

therfore $\forall x>0$, $x - \frac{x^2}{2} < \ln(1+x) < x$.

Exercise 03.

① The Maclaurin formula with the Lagrange remainder of $f(x)=e^x$ up to order n is:

$$f(x) = e^x = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$$

we have: $\begin{cases} f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1. \\ f^{(n+1)}(\theta x) = e^{\theta x} \end{cases}$

So:

$$f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!}x^{n+1}, \quad 0<\theta<1.$$

$$= \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{\theta x}}{(n+1)!}x^{n+1},$$

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② we show that : $1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!} < e < 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\frac{e}{(n+1)!}$

For $x=1$ in the previous formula, we obtain

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\frac{e^{\theta}}{(n+1)!}, \quad 0<\theta<1.$$

so:

$$0<\theta<1 \Rightarrow 1<e^{\theta}<e \Rightarrow \frac{1}{(n+1)!}<\frac{e^{\theta}}{(n+1)!}<\frac{e}{(n+1)!}$$

$$\Rightarrow 0<\frac{1}{(n+1)!}<e-\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)<\frac{e}{(n+1)!}.$$

$$\Rightarrow 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}<e<1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\frac{e}{(n+1)!}$$

③ Deduce the limit of the sequence $U_n=\sum_{k=0}^{n}\frac{1}{k!}=1+\frac{1}{1!}+\cdots+\frac{1}{n!}$

we have:

$$0<e-\sum_{k=0}^{n}\frac{1}{k!}<\frac{e}{(n+1)!} \Rightarrow 0<\lim_{n\to+\infty}\left(e-\sum_{k=0}^{n}\frac{1}{k!}\right)<\lim_{n\to+\infty}\frac{e}{(n+1)!}$$

then $\lim_{n\to+\infty}\left(e-\sum_{k=0}^{n}\frac{1}{k!}\right)=0$

so $\lim_{n\to+\infty}\sum_{k=0}^{n}\frac{1}{k!}=e.$

xercise 04

Find the Taylor-Young expansion of order 02:

① $f(x) = \ln x \quad , \quad x_0 = 1.$

we have $f(x) = f(x_0) + \dfrac{f'(x_0)}{1!}(x-x_0) + \dfrac{f''(x_0)}{2!}(x-x_0)^2 + O((x-x_0)^2)$

$= f(1) + \dfrac{f'(1)}{1!}(x-1) + \dfrac{f''(1)}{2!}(x-1)^2 + O((x-1)^2)$

$$\begin{cases} f(x) = \ln x \Longrightarrow f(1) = 0. \\ f'(x) = \dfrac{1}{x} \Longrightarrow f'(1) = 1 \\ f''(x) = \dfrac{-1}{x^2} \Longrightarrow f''(1) = -1. \end{cases}$$

So

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + O((x-1)^2).$$

② $f(x) = \sqrt{x} \quad , \quad x_0 = 4.$

we have: $f(x) = \sqrt{x} = f(4) + \dfrac{f'(4)}{1!}(x-4) + \dfrac{f''(4)}{2!}(x-4)^2 + O((x-4)^2)$

$$\begin{cases} f(x) = \sqrt{x} \Longrightarrow f(4) = 2. \\ f'(x) = \dfrac{1}{2\sqrt{x}} \Longrightarrow f'(4) = \dfrac{1}{4} \\ f''(x) = \dfrac{-1}{4x^{3/2}} \Longrightarrow f''(4) = \dfrac{-1}{32} \end{cases}$$

So $f(x) = \sqrt{x} = 2 + \dfrac{(x-4)}{4} - \dfrac{(x-4)^2}{64} + O((x-4)^2).$

③ $f(x) = \frac{1}{x}$, $x_0 = 1$.

we have: $f(x) = \frac{1}{x} = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + O((x-1)^2)$

$$\begin{cases} f(x) = \frac{1}{x} \Rightarrow f(1) = 1. \\ f'(x) = \frac{-1}{x^2} \Rightarrow f'(1) = -1 \\ f''(x) = \frac{2}{x^3} \Rightarrow f''(1) = 2 \end{cases}$$

so

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 + O((x-1)^2)$$

Exercise 05

• The Taylor-Young formula for $f(x) = \ln(x+1)$, $x_0 = 0$ and $n = 2$ is:

$$\ln(x+1) = x - \frac{x^2}{2} + O(x^2) \Rightarrow \ln(x+1) - x + \frac{x^2}{4} = -\frac{x^2}{4} + O(x^2)$$

• The Taylor-Young formula for $f(x) = \sin x$, $x_0 = 0$ and $n = 2$ is

$$\sin(x) = x + O(x^2) \Rightarrow (\sin x)^2 = x^2 + O(x^2).$$

$$\text{so } \lim_{x \to 0} \frac{\ln(x+1) - x + \frac{x^2}{4}}{(\sin x)^2} = \lim_{x \to 0} \frac{-\frac{x^2}{4} + O(x^2)}{x^2 + O(x^2)} = \frac{-1}{4}.$$

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Exercise 06.

① Calculate $LD_4(0)$ of $g \circ f$:

$$g \circ f(x) = g(f(x)) = 1 - \frac{1}{2}\left(x - \frac{x^3}{6}\right) + \frac{1}{24}\left(x - \frac{x^3}{6}\right)^4 + o(x^4)$$

$$= 1 - \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{x^4}{24} + o(x^4)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + o(x^4)$$

$$= 1 - \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^4)$$

②

$$\lim_{x \to 0} \frac{g \circ f(x) - 1 + \frac{x^2}{2}}{x^4}$$

$$= \lim_{x \to 0} \frac{\left(1 - \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^4)\right) - 1 + \frac{x^2}{2}}{x^4}$$

$$= \lim_{x \to 0} \frac{\frac{5x^4}{24} + o(x^4)}{x^4} = \frac{5}{24}.$$

Exercise 07

Calculate the $LD_n(x_0)$ of the following functions

① $f(x) = x\,(\cosh x)^{\frac{1}{x}}$, $x_0 = 0$, $n = 3$.

$$f(x) = x(\cosh x)^{\frac{1}{x}} = x\, e^{\frac{1}{x}\ln \cosh x}.$$

we have:

$$\cosh x = 1 + \frac{x^2}{2} + O(x^2)$$

so

$$\ln \cosh x = \ln\left(1 + \frac{x^2}{2} + O(x^2)\right)$$

we put $y = \frac{x^2}{2} + O(x^2)$, $x \longrightarrow 0$. $y \longrightarrow 0$.

Then:

$$\ln \cosh x = \ln(1+y) = y - \frac{y^2}{2} + O(y^2)$$

$$= \frac{x^2}{2} - \frac{x^4}{8} + O(x^4)$$

$$\Rightarrow \frac{1}{x}\ln \cosh x = \frac{x}{2} - \frac{x^3}{8} + O(x^3)$$

$$\Rightarrow e^{\frac{1}{x}\ln \cosh x} = e^{\frac{x}{2} - \frac{x^3}{8} + O(x^3)}$$

we put $z = \frac{x}{2} - \frac{x^3}{8} + O(x^3)$, $x \longrightarrow 0$, $z \longrightarrow 0$.

Then

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$$e^{\frac{1}{x}\ln\cosh x} = e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + o(z^3)$$

$$= 1 + \left(\frac{x}{2} - \frac{x^3}{8}\right) + \frac{1}{2}\left(\frac{x}{2} - \frac{x^3}{8}\right)^2 + \frac{1}{6}\left(\frac{x}{2} - \frac{x^3}{8}\right)^3 + o(x^3)$$

$$= 1 + \frac{x}{2} + \frac{x^2}{8} - \frac{5}{48}x^3 + o(x^3)$$

So

$$x e^{\frac{1}{x}\ln\cosh x} = x\left(1 + \frac{x}{2} + \frac{x^2}{8} - \frac{5}{48}x^3 + o(x^3)\right)$$

$$= x + \frac{x^2}{2} + \frac{x^3}{8} + o(x^3).$$

② $f(x) = \ln(1+\sin x)$ $\quad x_0 = 0 \quad n = 3.$

we have $\sin x = x - \frac{x^3}{3!} + o(x^3)$

so

$$\ln(1+\sin x) = \ln\left(1 + x - \frac{x^3}{3!} + o(x^3)\right)$$

we put $y = x - \frac{x^3}{3!} + o(x^3)$, $x \longrightarrow 0$, $y \longrightarrow 0$

So

$$\ln(1+\sin x) = \ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + o(y^3)$$

$$= \left(x - \frac{x^3}{6}\right) - \frac{1}{2}\left(x - \frac{x^3}{6}\right) + \frac{1}{3}\left(x - \frac{x^3}{6}\right)^3 + o(y^3)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} + o(x^3).$$

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③ $f(x) = \tan x = \dfrac{\sin x}{\cos x}$, $x_0 = 0$ $n = 5$

we have:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^5)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^5)$$

So:

$$\tan x = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^5)}{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^5)}$$

$$\begin{array}{l|l}
x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^5) & 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^5) \\ \cline{2-2}
x - \frac{x^3}{2} + \frac{x^5}{24} + o(x^5) & x + \frac{x^3}{3} + \frac{2}{15} x^5 \\ \cline{1-1}
\frac{x^3}{3} - \frac{x^5}{30} + O(x^5) & \\
\frac{x^3}{3} - \frac{x^5}{6} + O(x^5) & \\ \cline{1-1}
\frac{2x^5}{15} + O(x^5) & \\
\frac{2x^5}{15} + O(x^5) & \\ \cline{1-1}
O(x^5) &
\end{array}$$

So

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + O(x^5).$$

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4] $f(x) = \frac{\ln(1+x)}{1+x}$, $x_0 = 0$ $n=3$

$$= \ln(1+x) \times \frac{1}{1+x}$$

we have:

$$\begin{cases} \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \\ \frac{1}{1+x} = 1 - x + x^2 - x^3 + o(x^3) \end{cases}$$

So

$$\frac{\ln(1+x)}{1+x} = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\left(1 - x + x^2 - x^3\right) + o(x^3)$$

$$= x - \frac{3x^2}{2} + \frac{11}{6}x^3 + o(x^3).$$

⑤ $f(x) = e^{3x} \sin 2x$, $x_0 = 0$, $n = 4$

we have

$$\begin{cases} e^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + o(x^4) \\ \sin 2x = 2x - \frac{(2x)^3}{3!} + o(x^4). \end{cases}$$

So

$$e^{3x}\sin 2x = \left(1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!}\right)\left(2x - \frac{(2x)^3}{3!}\right) + o(x^4)$$

$$= 2x + 6x^2 + \frac{23}{3}x^3 + 5x^4 + o(x^4).$$

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⑥ $f(x) = e^{\sqrt{x}}$, $x_0 = 1$, $n=3$.

we put $t = x-1 \Rightarrow x = t+1$, $x \longrightarrow 1$, $t \longrightarrow 0$.

So $e^{\sqrt{x}} = e^{\sqrt{t+1}}$

we have:

$$\sqrt{t+1} = 1 + \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16} + o(t^3)$$

$$\Rightarrow e^{\sqrt{t+1}} = e^{1 + \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16} + o(t^3)}$$

we put $z = \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16} + o(t^3)$, $t \longrightarrow 0$, $z \longrightarrow 0$

So

$$e^{\sqrt{t+1}} = e^{1+z} = e \cdot e^{z} = e\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + o(z^3)\right)$$

$$= e\left(1 + \left(\frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16}\right) + \frac{1}{2}\left(\frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16}\right)^2 + \frac{1}{6}\left(\frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16}\right)^3 + o(t^3)\right)$$

$$= e\left(1 + \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16} + \frac{t^2}{8} - \frac{t^3}{16} + \frac{t^3}{48} + o(t^3)\right)$$

$$= e\left(1 + \frac{t}{2} + \frac{t^3}{48} + o(t^3)\right)$$

So

$$e^{\sqrt{x}} = e^{\sqrt{t+1}} = e\left(1 + \frac{t}{2} + \frac{t^3}{48} + o(t^3)\right) = e\left(1 + \frac{x-1}{2} + \frac{(x-1)^3}{48} + o\left((x-1)^3\right)\right)$$

Exercise 08.

① $\lim_{x \to 0} \frac{1}{\ln(1+x)} - \frac{1}{x}$

we have: $\ln(1+x) = x - \frac{x^2}{2} + O(x^2)$

so

$$\lim_{x \to 0} \frac{1}{\ln(1+x)} - \frac{1}{x} = \lim_{x \to 0} \left(\frac{1}{x - \frac{x^2}{2} + O(x^2)} - \frac{1}{x} \right)$$

$$= \lim_{x \to 0} \frac{1}{x} \left(\frac{1}{1 - \frac{x}{2} + O(x^2)} - 1 \right)$$

$$= \lim_{x \to 0} \frac{1}{x} \left(1 + \frac{x}{2} + O(x^2) - 1 \right) = \frac{1}{2}.$$

② $\lim_{x \to 0} \frac{e^{3x} \sin 3x}{\sinh(-2x)}$

we have:

$$\begin{cases} e^{3x} = 1 + 3x + O(x) \\ \sin 3x = 3x + O(x) \\ \sinh(-2x) = -2x + O(x). \end{cases}$$

So

$$\lim_{x\to 0} \frac{e^{3x}\sin 3x}{\sinh(-2x)} = \lim_{x\to 0} \frac{(1+3x+O(x))(3x+O(x))}{-2x+O(x)}$$

$$= \lim_{x\to 0} \frac{3x+O(x)}{-2x+O(x)} = -\frac{3}{2}.$$

③ $\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x$

we put $t = \frac{1}{x}$, $x \to +\infty$, $t \to 0$.

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{t\to 0}(1+t)^{\frac{1}{t}} = \lim_{t\to 0} e^{\frac{1}{t}\ln(1+t)}$$

$$= \lim_{t\to 0} e^{\frac{1}{t}\left(t-\frac{t^2}{2}+o(t^2)\right)} = \lim_{t\to 0} e^{1-\frac{t}{2}+o(t)}$$

$$= \lim_{t\to 0} e \cdot e^{-\frac{t}{2}+o(t)} = e.$$

④ $\lim_{x\to 0} \frac{x-\sin x}{x^3}$

we have: $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$

So

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{x-\left(x-\frac{x^3}{3}+\frac{x^5}{120}+o(x^5)\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{1}{6} - \frac{x^2}{120} + o(x^2) = \frac{1}{6}.$$

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(5) $\lim\limits_{x \to +\infty} x^2 \left(e^{\frac{1}{x}} - e^{\frac{1}{1+x}}\right)$

we put: $t = \frac{1}{x}$, $x \longrightarrow +\infty$, $t \longrightarrow 0$.

So: $\lim\limits_{x \to +\infty} x^2 \left(e^{\frac{1}{x}} - e^{\frac{1}{x+1}}\right) = \lim\limits_{t \to 0} \frac{1}{t^2} \left(e^t - e^{\frac{t}{1+t}}\right)$

Since:

$$\begin{cases} \dfrac{t}{1+t} = t - t^2 + o(t^2) \\ e^t = 1 + t + \dfrac{t^2}{2} + o(t^2) \end{cases}$$

So

$$e^{\frac{t}{1+t}} = 1 + (t - t^2) + \frac{(t-t^2)^2}{2} + o(t^2)$$

$$= 1 + t - \frac{t^2}{2} + o(t^2).$$

then:

$$\lim_{x \to +\infty} x^2 \left(e^{\frac{1}{x}} - e^{\frac{1}{x+1}}\right) = \lim_{t \to 0} \frac{1}{t^2} \left(e^t - e^{\frac{t}{1+t}}\right)$$

$$= \lim_{t \to 0} \frac{1}{t^2}\left(t^2 + o(t^2)\right) = 1.$$

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