University of Mila
Institute of mathematics and computer science
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Department of informatics
Analysis 2 (1 st Year)

Series of Exercises N°1

Exercise 1

Using Taylor-Lagrange formula, prove that:

❶ $\forall x \in \left[0, \frac{\pi}{2}\right], \quad x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}.$

❷ $\forall x \in \mathbb{R}_+, \quad 1 + \frac{x}{3} - \frac{x^2}{9} \leq \sqrt[3]{1+x} \leq 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{91}.$ (*)

Exercise 2

Using the Maclaurin expansion of the function $\ln(1+x)$, show that

$$\forall x > 0, \quad x - \frac{x^2}{2} < \ln(1+x) < x.$$

Exercise 3

❶ Find the Maclaurin formula with the Lagrange remainder of $f(x) = e^x$ up to order n .

❷ Show that: $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e}{(n+1)!}.$

❸ Deduce the limit of the sequence

$$u_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

Exercise 4

Find the Taylor-Young expansion of order 2 near the given point.

❶ $f(x) = \ln x, \quad x_0 = 1$

❷ $f(x) = \sqrt{x}, \quad x_0 = 4.$

❸ $f(x) = \frac{1}{x}, \quad x_0 = 1.$

Exercise 5

Using the Taylor-Young formula, calculate the following limit

$$\lim_{x\to 0} \frac{\ln(x+1) - x + \frac{x^2}{4}}{(\sin x)^2}$$

Exercise 6

Let f and g be two functions that admit limited developments in a neighborhood of 0 given by

$$f(x) = x - \frac{x^3}{6} + O(x^4)$$

$$g(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^4)$$

❶ Calculate the limited development up to order 4 in a neighborhood of zero of the composite function gof.

❷ Deduce the following limit

$$\lim_{x\to 0} \frac{gof(x) - 1 + \frac{x^2}{2}}{x^4}$$

Exercise 7

Compute the limited developments up to order n in the neighborhood of x_0 of the following functions:

❶ $f(x) = x(\cosh x)^{\frac{1}{x}}, \quad x_0 = 0, \quad n = 3.$

❷ $f(x) = \ln(1 + \sin x), \quad x_0 = 0, \quad n = 3.$

❸ $f(x) = \tan(x), \quad x_0 = 0, \quad n = 5.$

❹ $f(x) = \frac{\ln(1+x)}{1+x}, \quad x_0 = 0, \quad n = 3.$

❺ $f(x) = e^{3x}\sin(2x), \quad x_0 = 0, \quad n = 4.$

❻ $f(x) = e^{\sqrt{x}}, \quad x_0 = 1, \quad n = 3.$

Exercise 8

Calculate the following limits:

❶ $\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)$

❷ $\lim_{x\to 0}\frac{e^{3x}\sin 3x}{\sinh(-2x)}$

❸ $\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^{x}$

❹ $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

❺ $\lim_{x\to +\infty} x^2\left(e^{1/x}-e^{1/(1+x)}\right)$