Chapter 03: Indefinite integral

1/ Indefinite integral

Definition.

The set of all primitives of the function $f: I \longrightarrow \mathbb{R}$ is called the indefinite integral of f and denoted by $\int f(x)\,dx$.

$$\int f(x)\,dx = F(x) + c, \quad c \in \mathbb{R}.$$

Example.

- $\int \sin x\,dx = -\cos x + c, \quad c \in \mathbb{R}$

- $\int \frac{1}{x}\,dx = \ln|x| + c, \quad c \in \mathbb{R}.$

- $\int 5x^2 + 6\,dx = 5\frac{x^3}{3} + 6x + c, \quad c \in \mathbb{R}$

2/ Integration by parts and change of variable in indefinite integrals.

* Integration by parts.

Theorem.

Let $f, g : I \longrightarrow \mathbb{R}$, $f, g \in C^1(I)$, then.

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx.$$

Example.

- Compute $\int x\, e^x\, dx$.

we have:

$$\begin{cases} f(x) = x \\ g'(x) = e^x \end{cases} \Longrightarrow \begin{cases} f'(x) = 1 \\ g(x) = e^x \end{cases}$$

So

$$\int x e^x = x e^x - \int e^x dx = x e^x - e^x + c = e^x(x-1) + c, \ c \in \mathbb{R}.$$

- Compute $\int x \ln x\, dx$.

we have :

$$\begin{cases} f(x) = \ln x \\ \\ g'(x) = x \end{cases} \Longrightarrow \quad \begin{matrix} f'(x) = \frac{1}{x} \\ \\ g(x) = \frac{x^2}{2} \end{matrix}$$

so

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + C, \quad C \in \mathbb{R}.$$

* Change of variable.

Let f be a continuous function on I and $g \in C^1(I)$

If $x = g(t) \Rightarrow dx = g'(t)\, dt$, then :

$$\int f(x)\, dx = \int f(g(t)) \cdot g'(t)\, dt = F(g(t)).$$

Example.

Calculate $I = \int \frac{1}{x + \sqrt{x}} dx.$

we put $t = \sqrt{x} \Rightarrow x = t^2 \Rightarrow dx = 2t\, dt.$

so $\int \frac{1}{x+\sqrt{x}} dx = \int \frac{1}{t^2+t} 2t\, dt = 2\int \frac{1}{t+1} dt = 2 \ln|t+1| + C = 2\ln(\sqrt{x}+1) + C$

$C \in \mathbb{R}$

3/ Integration of rational functions in e^x.

This type of integration can always be reduced to a rational fraction by setting $t = e^x$.

Example.

Compute $I = \int \frac{1}{1+e^x} \, dx$.

we put $t = e^x \Rightarrow x = \ln t \Rightarrow dx = \frac{dt}{t}$.

So

$$\int \frac{1}{1+e^x} dx = \int \frac{1}{1+t} \cdot \frac{dt}{t} = \int \left(\frac{a}{1+t} + \frac{b}{t} \right) dt$$

$$= \int \frac{-1}{1+t} dt + \int \frac{1}{t} dt$$

$$= -\ln|1+t| + \ln|t| + c, \quad c \in \mathbb{R}$$

$$= \ln\left|\frac{t}{1+t}\right| + c, \quad c \in \mathbb{R}$$

$$= \ln\left|\frac{e^x}{1+e^x}\right| + c, \quad c \in \mathbb{R}.$$

4/ Integration of rational functions of $\sin x$ and $\cos x$

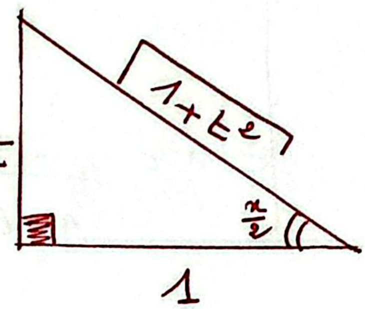
we distinguish the following cases:

① Integrals of type $\int R(\cos(x), \sin x)\, dx$

with $R(x)$ is a rational fraction.

we put $t = \tan\left(\frac{x}{2}\right)$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$$dx = \frac{2}{1+t^2}\, dt.$$



• $t = \tan\left(\frac{x}{2}\right) = \frac{\text{المقابل}}{\text{المجاور}}$.

$$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$$

$$\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$$

$$\begin{cases} \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right) = \frac{2t}{1+t^2} \\ \cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 = \frac{1-t^2}{1+t^2} \end{cases}$$

$$t = \tan\left(\frac{x}{2}\right) \Rightarrow \frac{x}{2} = \arctan t \Rightarrow x = 2\arctan t$$

$$\Rightarrow dx = 2.(\arctan t)'\, dt$$

$$\Rightarrow dx = \frac{2}{1+t^2}\, dt.$$

Example compute $I = \int \frac{1}{1-\cos x}\, dx.$

we put $\begin{cases} t = \tan \frac{x}{2} \\ \cos x = \frac{1-t^2}{1+t^2} \\ dx = \frac{2}{1+t^2}\, dt \end{cases}$

so

$$I = \int \frac{1}{1-\cos x}\, dx = \int \frac{1}{1 - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$$

$$= \int \frac{\cancel{1+t^2}}{\cancel{2}t^2} \cdot \frac{\cancel{2}}{\cancel{1+t^2}}\, dt = \int \frac{1}{t^2}\, dt = -\frac{1}{t} + C,\ C \in \mathbb{R}$$

$$= -\frac{1}{\tan\left(\frac{x}{2}\right)} + C,\ C \in \mathbb{R}$$

② Integrals of type $\int R(\cos(x)) \times \sin x \, dx$.

with $R(x)$ is a rational fraction:

we put $t = \cos x$ and $dt = -\sin x \, dx$.

③ Integrals of type $\int R(\sin(x)) \times \cos x \, dx$.

we put $t = \sin x$ and $dt = \cos x \, dx$

Examples.

① Compute $I = \int \frac{\cos^3 x}{\sin^2 x} dx$.

we have $\frac{\cos^3 x}{\sin^2 x} = \frac{\cos^2 x \cos x}{\sin^2 x} = \frac{1 - \sin^2 x}{\sin^2 x} \cos x$

so I of type $\int R(\sin(x)) \cos x \, dx$.

we put $t = \sin x \Rightarrow dt = \cos x \, dx$.

so: $I = \int \frac{1 - \sin^2 x}{\sin^2 x} \cos x \, dx = \int \frac{1 - t^2}{t^2} dt = \int \left(\frac{1}{t^2} - 1\right) dt$

$= -1/t - t + C = -1/\sin x - \sin x + C, \quad C \in \mathbb{R}$.

② compute $I = \int \frac{\sin^3 x}{\cos x} dx$.

we have $\frac{\sin^3 x}{\cos x} = \frac{\sin^2 x}{\cos x} \times \sin x = \frac{1-\cos^2 x}{\cos x} \times \sin x$

so I of type $\int R(\cos(x)) \times \sin x \, dx$.

we put $t = \cos x \Rightarrow dt = -\sin x \, dx$.

So: $I = \int \frac{1-\cos^2 x}{\cos x} \times \sin x \, dx = -\int \frac{1-t^2}{t} dt$

$= -\int \left(\frac{1}{t} - t\right) dt$

$= -\ln|t| + \frac{t^2}{2} + C, \quad C \in \mathbb{R}.$

$= -\ln|\cos x| + \frac{\cos^2 x}{2} + C, \quad C \in \mathbb{R}.$

5 Primitive of rational functions

In this section we study how to compute the primitive of a rational function of the form : $f(x) = \dfrac{P(x)}{q(x)}$

where p and q are polynomials with real coefficients.

we distinguish two cases:

1. If $\deg P \geq \deg q$, we perform a Euclidean division according to the decreasing powers of x.

Thus:

$$f(x) = \frac{P(x)}{q(x)} = S(x) + \frac{R(x)}{q(x)}.$$

where $S(x)$ and $R(x)$ are polynomials and $\deg R < \deg q$

2. If $\deg P \leq \deg q$, the rational function $\frac{P(x)}{q(x)}$ can be decomposed into partial fractions.

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• First we focus on the primitive of rational functions of the form:

$$f(x) = \frac{Ax + B}{ax^2 + bx + c}.$$

(1) we factorize $ax^2 + bx + c$ by computing the discriminant $\Delta = b^2 - 4ac$.

• if $\Delta > 0$, then: $ax^2 + bx + c = a(x - x_1)(x - x_2)$

• if $\Delta = 0$, then $ax^2 + bx + c = a(x - x_1)^2$.

• if $\Delta < 0$ then $ax^2 + bx + c$ has no real roots and therefore cannot be factored over $\mathbb{R}$.

Example

Compute $I = \int \frac{5x - 4}{2x^2 + x - 1}\, dx.$

1) we factorize $2x^2 + x - 1$.

$\Delta = b^2 - 4ac = (1)^2 - 4(2)(-1) = 9 > 0.$

so $x_1 = \frac{-b - \sqrt{\Delta}}{2a} = -1$, $x_2 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{1}{2}.$

So $2x^2 + x - 1 = 2(x+1)\left(x - \frac{1}{2}\right) = (x+1)(2x-1)$.

Then $\dfrac{5x-4}{2x^2+x-1} = \dfrac{5x-4}{(x+1)(2x-1)}$

②We determine real numbers a and b such that:

$$\frac{5x-4}{(x+1)(2x-1)} = \frac{a}{(x+1)} + \frac{b}{(2x-1)}$$

$$\frac{5x-4}{(x+1)(2x-1)} = \frac{a}{x+1} + \frac{b}{(2x-1)}$$

$$= \frac{a(2x-1) + b(x+1)}{(x+1)(2x-1)}$$

$$= \frac{2ax - a + bx + b}{(x+1)(2x-1)} = \frac{(2a+b)x - a + b}{(x+1)(2x-1)}$$

so:

$$\begin{cases} 2a + b = 5 \\ -a + b = -4 \end{cases}$$

Then: $\boxed{a = 3}$ and $\boxed{b = -1}$

so: $\dfrac{5x-4}{(x+1)(2x-1)} = \dfrac{3}{x+1} - \dfrac{1}{2x-1}$.

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3. we Calculate the integral.

$$I = \int \frac{5x-4}{2x^2+x-1}\,dx = \int \frac{3}{x+1}\,dx - \int \frac{1}{2x-1}\,dx.$$

$$= 3\ln|x+1| - \frac{1}{2}\ln|2x-1| + c,\ c \in \mathbb{R}.$$

Example.

Compute $I = \int \frac{7x-4}{x^2-2x+1}\,dx$

1 we factorize x^2-2x+1

$$\Delta = b^2-4ac = (-2)^2-4(1)(1) = 0.$$

so $x_1 = x_2 = \frac{-b}{2a} = \frac{+2}{2} = +1.$

Then $x^2-2x+1 = (x-1)^2$ so: $\frac{7x-4}{x^2-2x+1} = \frac{7x-4}{(x-1)^2}.$

2 we Determine real numbers a and b such that

$$\frac{7x-4}{(x-1)^2} = \frac{a}{(x-1)} + \frac{b}{(x-1)^2}.$$

$$\frac{7x-4}{(x-1)^2} = \frac{a}{(x-1)} + \frac{b}{(x-1)^2} = \frac{a(x-1)+b}{(x-1)^2} = \frac{ax-a+b}{(x-1)^2}$$

so $\boxed{a=7}$ and $\boxed{b=3}$

Then $\dfrac{7x-4}{(x-1)^2} = \dfrac{7}{(x-1)} + \dfrac{3}{(x-1)^2}$.

③ we calculate the integral.

$$I = \int \frac{7x-4}{x^2-2x+1}\,dx = 7\int \frac{1}{x-1}\,dx + 3\int \frac{1}{(x-1)^2}\,dx$$

$$= 7\ln|x-1| - \frac{3}{x-1} + C \quad , \quad C \in \mathbb{R}.$$

Example

Compute $I = \displaystyle\int \frac{1}{x^2+2x+5}\,dx$.

we have $\Delta = b^2 - 4ac = (2)^2 - 4(1)(5) = 4 - 20 = -16 < 0$.

① we will try to write x^2+2x+5 in the form

$a(t^2+1)$.

$$x^2+2x+5 = x^2+2x+1+4 = (x+1)^2+4$$

$$= 4\left(\left(\frac{x+1}{2}\right)^2+1\right)$$

So $\displaystyle\int \frac{1}{x^2+2x+5}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x+1}{2}\right)^2+1}\,dx.$

② we put $t = \frac{x+1}{2} \Rightarrow x = 2t - 1 \Rightarrow dx = 2dt$

SO :

$$I = \frac{1}{4}\int \frac{1}{t^2+1}\, 2dt = \frac{1}{2}\int \frac{1}{t^2+1}\, dt$$

$$= \frac{1}{2} \arctan t + c, \quad c \in \mathbb{R}$$

$$= \frac{1}{2} \arctan\left(\frac{x+1}{2}\right) + c, \quad c \in \mathbb{R}.$$

Example.

Compute $I = \int \frac{x^2+3x-1}{x-1}\, dx.$

we have $\frac{x^2+3x-1}{x-1} = (x+4) + \frac{3}{x-1}$. (Euclidean division)

SO

$$\int \frac{x^2+3x-1}{x-1}\, dx = \int (x+4)\, dx + \int \frac{3}{x-1}\, dx$$

$$= \frac{x^2}{2} + 4x + 3 \ln|x-1| + c, \quad c \in \mathbb{R}.$$

primitive of the form $I_n = \int \frac{1}{(x^2+1)^n} dx, n \in \mathbb{N}$

by part, we have:

$$\begin{cases} f(x) = \dfrac{1}{(x^2+1)^n} \\ g'(x) = 1 \end{cases} \Longrightarrow \quad f'(x) = \frac{-2nx}{(x^2+1)^{n+1}} \qquad g(x) = x.$$

SO

$$I_n = \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2}{(x^2+1)^{n+1}} dx.$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2+1-1}{(x^2+1)^{n+1}} dx.$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2+1}{(x^2+1)^{n+1}} dx - 2n \int \frac{1}{(x^2+1)^{n+1}} dx$$

$$= \frac{x}{(x^2+1)^n} + 2n \int \frac{1}{(x^2+1)^n} dx - 2n \int \frac{1}{(x^2+1)^{n+1}} dx$$

$$= \frac{x}{(x^2+1)^n} + 2n I_n - 2n I_{n+1}$$

Then $I_{n+1} = \frac{1}{2n}\left[(2n-1) I_n + \frac{x}{(x^2+1)^n}\right].$

where $I_1 = \int \frac{1}{x^2+1} \, dx = \arctan x + C, \; C \in \mathbb{R}.$

Example.

Compute $I_2 = \int \frac{1}{(x^2+1)^2} \, dx.$

• We have: $I_1 = \int \frac{1}{(x^2+1)} \, dx = \arctan x + C.$

$$I_{n+1} = \frac{1}{2n}\left[(2n-1) I_n + \frac{x}{(x^2+1)^n}\right].$$

SO

$$I_2 = \int \frac{1}{(x^2+1)^2} \, dx = \frac{1}{2}\left[I_1 + \frac{x}{(x^2+1)}\right]$$

$$= \frac{1}{2} \arctan x + \frac{x}{2(x^2+1)} + C, \quad C \in \mathbb{R}.$$