Chapter 02: The definite Integrals.

2.1. Partition

Definition.

let f be a function defined and bounded on $[a,b]$

A subdivision (or partition) of $[a,b]$ is any finite set of points in $[a,b]$ that contains a and b.

$$P = \{x_0, x_1, x_2, \ldots x_n\} \subset [a,b] \text{ where } a = x_0 < x_1 < x_2 \cdots < x_n = b.$$

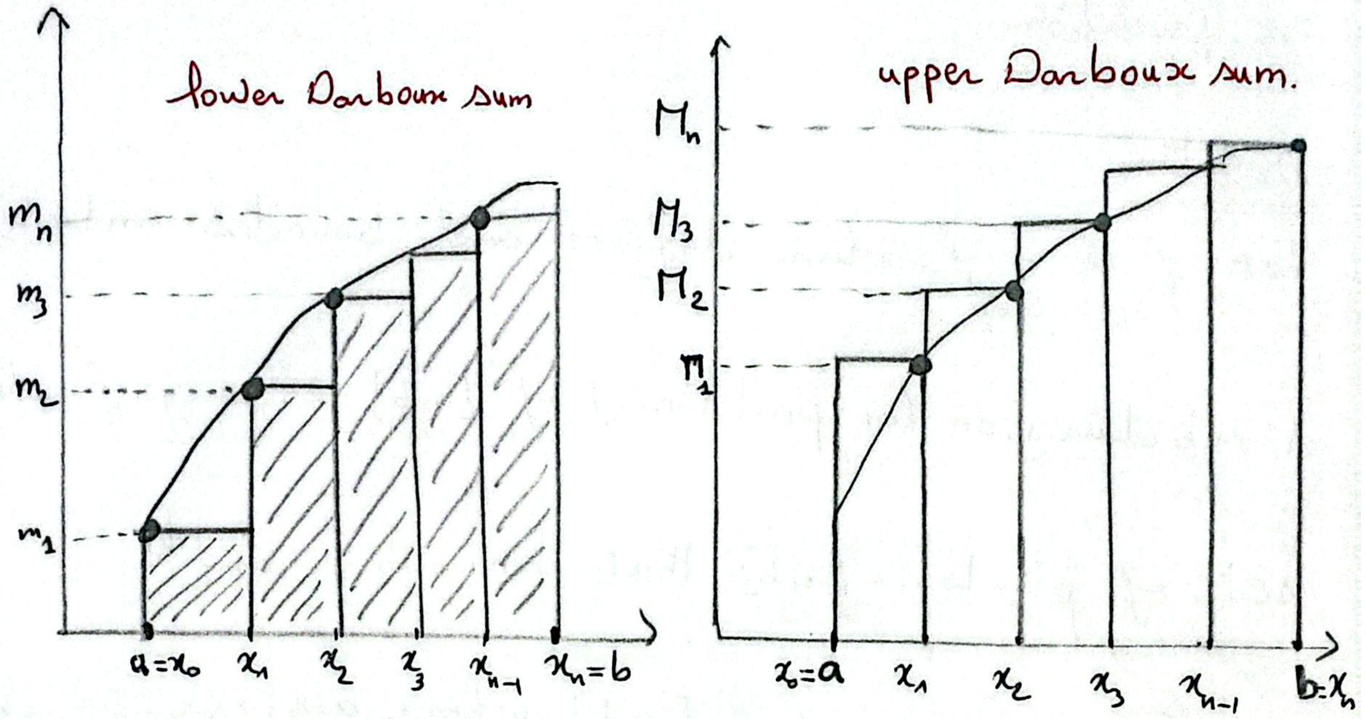
- The division of the interval $[a,b]$ by the partition P generates n subinterval:

$$[x_0, x_1]\cdot [x_1, x_2]\ [x_2, x_3] \ldots [x_{n-1}, x_n].$$

- The length of each interval $[x_{k-1}, x_k]$ is $\Delta x_k = x_k - x_{k-1}$
- The union of subinterval gives the whole interval $[a,b]$
- The strictly positive real number $\delta(P) = \max (x_k - x_{k-1})$ is the maximum length of a subintervals is called the mesh of the partition

2-2 Darboux sums

Definition



Let f be a function defined and bounded on $[a,b]$

and P is a partition of $[a,b]$ and let

$$m_k = \inf_{x_{k-1} < x < x_k} f(x)$$

$$M_k = \sup_{x_{k-1} < x < x_k} f(x)$$

The lower and upper Darboux sums of f

corresponding to a partition P:

$$L(f,P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

$$U(f,P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1}).$$

2.3) Upper and lower Integrals.

Definition

Suppose $f:[a,b] \longrightarrow \mathbb{R}$ is bounded. Then we define

the lower integral of f on $[a,b]$ as:

$$\int_{*a}^{b} f(x)\,dx = \sup(L) \text{ where } L = \{L(f,P),\ P \text{ is a partition of } [a,b]\}$$

and the upper integral of f on $[a,b]$ as:

$$\int_{a}^{*b} f(x)\,dx = \inf(U) \text{ where } U = \{U(f,P),\ P \text{ is a partition of } [a,b]\}$$

2.4) properties of Darboux sums.

- For all partition $P \subset [a,b]$ we have $U(f,P) \geq L(f,P)$

- If $P \subset P'$ then $L(f,P) \leq L(f,P')$ and $U(f,P) \geq U(f,P')$.

- $\forall\ P, P' \subset [a,b]$

$$L(f,P) \leq \int_{*a}^{b} f(x)\,dx \leq \int_{a}^{b} f(x)\,dx \leq \int_{a}^{*b} f(x)\,dx \leq U(f,P')$$

2.5) Integrable functions, Riemann integrals.

Definition

Suppose $f : [a,b] \longrightarrow \mathbb{R}$ is Bounded, we say that f is integrable in the sense of Riemann if

$$\int_{*a}^{b} f(x)\,dx = \int_{a}^{*b} f(x)\,dx.$$

The value of the lower integral and the upper integral is called the Riemann integral of f over $[a,b]$. and is denoted by $\int_a^b f(x)dx$.

Theorem.

A function f is Riemann integrable on $[a,b]$ iff:

$$\forall \varepsilon > 0, \exists P \subset [a,b], \quad U(f,P) - L(f,P) < \varepsilon.$$

Theorem.

- Any bounded and monotonic function on $[a,b]$ is integrable on $[a,b]$.
- Any continuous function on $[a,b]$ is integrable on $[a,b]$

2-6/ Riemann Sums

Definition

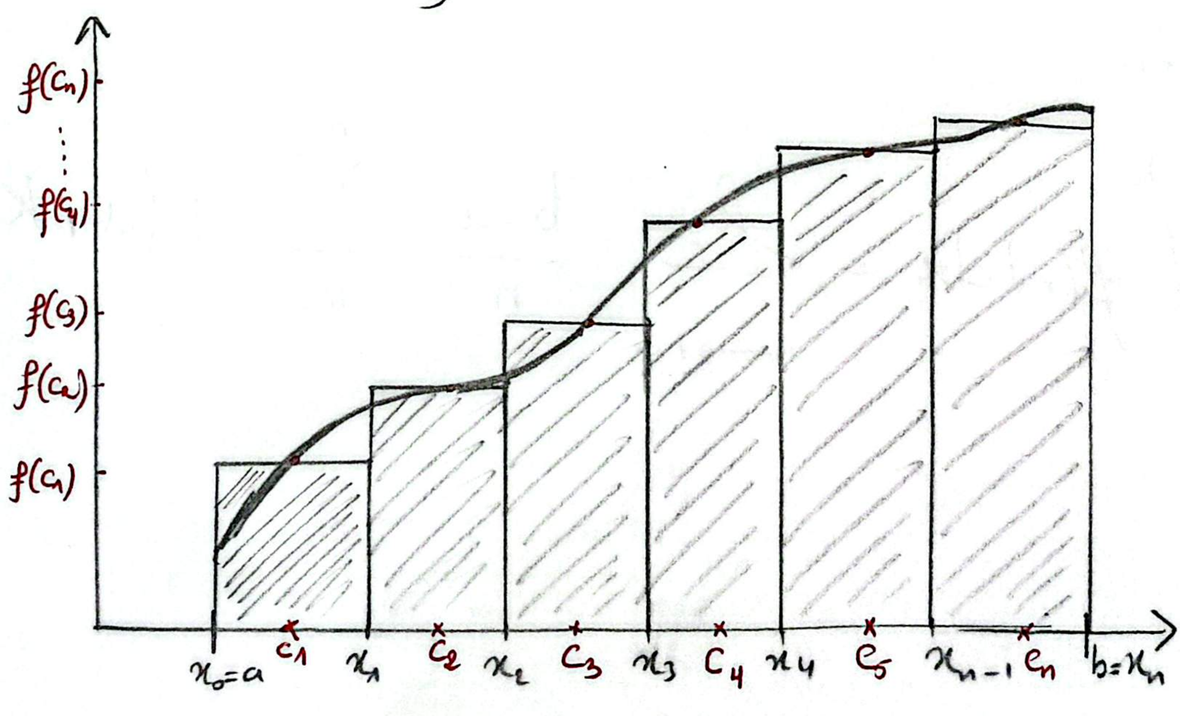
Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \ldots, x_n\}$ a partition of $[a,b]$

The sum :

$$R(f,P) = \sum_{k=1}^{n} (x_k - x_{k-1})\, f(c_k) \text{ where } c_k \in [x_{k-1}, x_k]$$

is called the Riemann sum of f corresponding to P and the system of points

$c = (c_1, c_2, \ldots, c_n)$



Riemann sums

Theorem.

If $f : [a,b] \longrightarrow \mathbb{R}$ is integrable then

$$\int_a^b f(x)\,dx = \lim_{\delta(P)\to 0} R(f,P)$$

particular case.

For a regular partition of $[a,b]$ into n equal subintervals, we have:

$$\Delta x = \frac{b-a}{n} \qquad x_k = a + k\,\frac{b-a}{n}.$$

Then.

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)$$

Example. Using the Riemann sums calculate $\int_1^2 x\,dx$

$f(x) = x$, $a = 1$, $b = 2$.

we have. $$\int_1^2 x\,dx = \lim_{n\to+\infty} \frac{b-a}{n}\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)$$

$$= \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(1+\frac{k}{n}\right)$$

$$= \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \left(1+\frac{k}{n}\right)$$

$$= \lim_{n\to+\infty} \left(\frac{1}{n}\sum_{k=1}^{n} 1\right) + \left(\frac{1}{n^2}\sum_{k=1}^{n} k\right)$$

$$= \lim_{n\to+\infty} \frac{n}{n} + \frac{1}{n^2}\left(\frac{n}{2}(1+n)\right)$$

$$= \lim_{n\to+\infty} 1 + \frac{n+1}{2n}$$

$$= \frac{3}{2}.$$

2.7/ properties of the Riemann integral

Let f and g be two bounded and integrable functions on $[a,b]$, $c\in[a,b]$ and $\alpha\in\mathbb{R}$.

① $\int_a^a f(x)\,dx = 0$.

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

③ $\int_a^b \left(f(x) + g(x)\right) dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

④ $\int_a^b \left(\alpha f(x)\right) dx = \alpha \int_a^b f(x)\,dx.$

⑤ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

⑥ $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$

⑦ If $f(x) \leq g(x) \implies \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$

⑧ If $f(x) \geq 0 \implies \int_a^b f(x)\,dx \geq 0.$

⑨ If $f \geq 0$ and continuous on $[a,b]$ and if $\int_a^b f(x)\,dx = 0$

$\implies f(x) = 0$ on $[a,b]$.

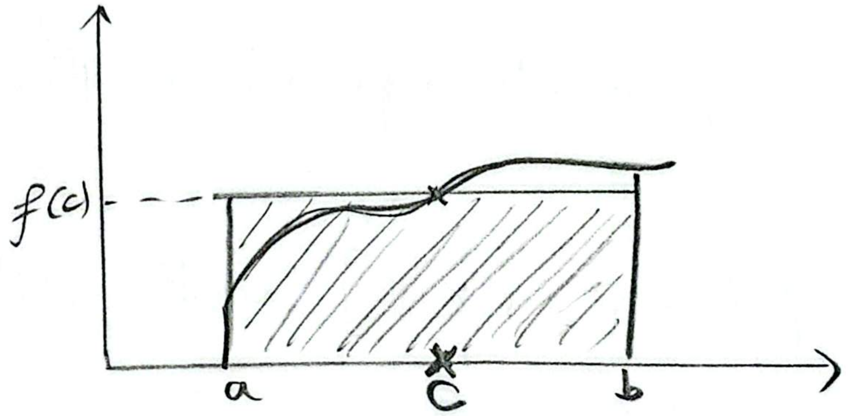
2.8/ Mean value theorem for integrals

If f is continuous function on $[a,b]$

then there exists $c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)\,(b-a).$$

The number $f(c)$ is called the average value of f over $[a,b]$.



2.9) Cauchy - Schwarz Inequality.

If f and g are two bounded and integrable functions on $[a,b]$, we have

$$\left(\int_a^b f(x)\,g(x)\,dx\right)^2 \leqslant \int_a^b f^2(x)\,dx \times \int_a^b g^2(x)\,dx.$$

10/ primitive of a continuous functions.

Definition

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function, we call the primitive of f any differentiable function $F : [a,b] \longrightarrow \mathbb{R}$ such that

$$F'(x) = f(x) \qquad \forall x \in [a,b].$$

Theorem.

If a function f admits a primitive F on $[a,b]$ then the set $\{F + c,\ c \in \mathbb{R}\}$ is the set of all primitives of f on $[a,b]$.

Example.

① $f(x) = x^2 + 5 \implies F(x) = \frac{x^3}{3} + 5x + c,\quad c \in \mathbb{R}$

② $f(x) = x^3 + \cos x \implies F(x) = \frac{x^4}{4} + \sin x + c,\quad c \in \mathbb{R}.$

Theorem.

Every continuous function on $[a,b]$ admits a primitive on $[a,b]$.

Definition.

Let f be a continuous function on $[a,b]$ and F a primitive of f, then

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Example.

- $\displaystyle\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$

- $\displaystyle\int_0^{\frac{\pi}{2}} \cos x\,dx = \left[\sin x\right]_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1.$

2.11/ Integration methods.

- **Direct integration.**

Table of usual primitives.

$f(x)$	$F(x)$
a (constant)	$ax + c$
$x^n,\ n \geq 1$	$\frac{x^{n+1}}{n+1} + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
$\frac{1}{x^n},\ n \geq 2$	$\frac{-1}{(n-1)\, x^{n-1}} + c$
$\ln x$	$x \ln x - x + c$
e^x	$e^x + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\cosh x$	$\sinh x + c$
$\sinh x$	$\cosh x + c$
$\frac{1}{1+x^2}$	$\arctan x + c.$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$

• Integration by parts

Let f and g be two differentiable functions on $[a,b]$, then we have:

$$\int_a^b f'(x)\, g(x)\, dx = \Big[f(x)\, g(x)\Big]_a^b - \int_a^b f(x)\, g'(x)\, dx.$$

This result is a direct consequence of the derivative of the product of two functions.

Example.

① Calculate $\int_0^1 x e^x dx$.

we put:

$$\begin{cases} f(x) = x \\ g'(x) = e^x \end{cases} \implies \begin{array}{l} f'(x) = 1 \\ g(x) = e^x \end{array}$$

then

$$\int_0^1 e^x dx = \Big[x e^x\Big]_0^1 - \int_0^1 e^x = e - \Big[e^x\Big]_0^1 = 1.$$

② $\int_1^e \ln x \, dx$.

we have:

$$\begin{cases} f(x) = \ln x & f'(x) = \frac{1}{x} \\ g'(x) = 1 & g(x) = x \end{cases}$$

so

$$\int_1^2 \ln x \, dx = \left[x \ln x\right]_1^2 - \int_1^2 1 \, dx.$$

$$= 2\ln 2 - \left[x\right]_1^2$$

$$= 2\ln 2 - 1.$$

• Integration By change of variables.

Let f be a continuous function on $[a,b]$ and $g \in C^1([a,b])$

This method consists of putting $x = g(t)$. , in the integral and $dx = g'(t)\,dt$.

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(t))\,g'(t)\,dt.$$

Example

Calculate $I = \int_1^4 \frac{1}{x+\sqrt{x}}\,dx$.

we put : $t = \sqrt{x} \Rightarrow x = t^2 \Rightarrow dx = 2t\,dt$.

$$\begin{cases} \text{if } x = 1 \Rightarrow t = \sqrt{1} = 1. \\ \text{if } x = 4 \Rightarrow t = \sqrt{4} = 2 \end{cases} \Rightarrow \int_1^4 dx = \int_1^2 dt.$$

SO :

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx = \int_1^2 \frac{1}{t^2+t}\,2t\,dt = 2\int_1^2 \frac{1}{t+1}\,dt$$

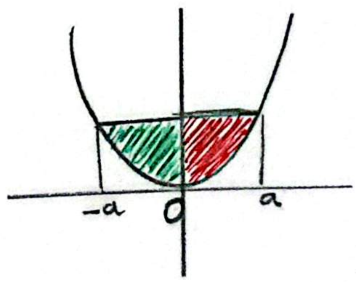
$$= 2\Big[\ln|t+1|\Big]_1^2 = 2\big[\ln 3 - \ln 2\big] = 2\ln\frac{3}{2}.$$

proposition.

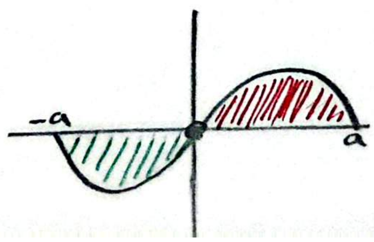
Let f be a continuous function on $[a, b]$, then

① If f is an even function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$



② If f is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$.



Operations on primitive.

Let u be a differentiable function on I so.

- A primitive of $u'u^n$ on I is $\dfrac{u^{n+1}}{n+1}$ $(n \in \mathbb{N}^*)$

- " " $\dfrac{u'}{u^2}$ on I is $\dfrac{-1}{u}$.

- " " $\dfrac{u'}{u^n}$ on I is $-\dfrac{1}{(n-1)\,u^{n-1}}$ $(n \geq 2)$

- " " $\dfrac{u'}{\sqrt{u}}$ on I is $2\sqrt{u}$. $(u > 0)$

- " " $\dfrac{u'}{u}$ " " $\ln|u|$.

- " " $u'e^u$ " " e^u.