Chapter 01 : Limited development

1/ Taylor's formulas

Definition

A function that is continuous on $[a,b]$ and differentiable on $x_0 \in]a,b[$ can be expressed in a neighborhood of x_0 as follows:

$$f(x) = f(x_0) + (x - x_0)\, f'(x_0) + R(x)$$

where $R(x)$ is the remainder of order 1.

This shows that if f is differentiable then f can be approximated by a polynomial of degree 1 (a line).

Example.

Consider the function $f(x) = e^x$ and $x_0 = 0$ so f can be written as: $f(x) \simeq f(0) + (x - 0) f'(0) = 1 + x$.

Taylor's formula generalizes this result by showing that functions that are n-times differentiable can be approximated in a neighborhood of x_0 by polynomials of degree n.

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + (x - x_0)^2 \frac{f''(x_0)}{2!}$$

$$+ \cdots + (x - x_0)^n \frac{f^{(n)}(x_0)}{n!} + R_n(x).$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_n(x)$$

where $R_n(x)$ is the remainder of order n.

2/ Taylor's three formulas.

* Taylor-Lagrange.

Theorem

Let f be of class C^{n+1} and let $x_0 \in [a,b]$

For all $x \in [a,b]$, $x \neq x_0$, we have:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$$+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

where $c \in]x_0, x[$.

The term $\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ is called the lagrange remainder.

Example.

1/ Consider the function $\sin x$. The taylor-Lagrange formula up to order 3 in the neighborhood of 0 is written as follows:

$$\sin(x) = \sin(0) + \frac{\sin'(0)}{1!}(x-0) + \frac{\sin''(0)}{2!}(x-0)^2 + \frac{\sin'''(0)}{3!}(x-0)^3$$

$$+ \frac{\sin^{(4)}(c)}{4!}(x-0)^4 = x - \frac{x^3}{6} + \frac{x^4}{24}\sin(c) \qquad c \in]0,x[$$

2) Consider the function e^x. The Taylor-Lagrange formula up to order 4 in the neighborhood of 0 is:

$$e^x = e^0 + \frac{e^0}{1!}(x-0) + \frac{e^0}{2!}(x-0)^2 + \frac{e^0}{3!}(x-0)^3$$

$$+ \frac{e^0}{4!}(x-0)^4 + \frac{e^c}{5!}(x-0)^5 \qquad c \in]0,x[$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{e^c}{5!}x^5.$$

Remark.

If $x_0 = 0$, we obtain the Maclaurin-Lagrange formula.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} + \dots + \frac{f^{(n)}(0)x^n}{n!} + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$$

where $\theta \in]0,1[$.

* Taylor-young

Theorem.

Let f be of class $C^n([a,b])$ and let $x_0 \in [a,b]$ for all $x \in [a,b]$, $x \neq x_0$, we have:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$$+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + O\left((x-x_0)^n\right)$$

where $O((x-x_0)^n) = (x-x_0)^n \varepsilon(x)$ and $\lim\limits_{x \to x_0} \varepsilon(x) = 0$.

The term $(x-x_0)^n \varepsilon(x)$ is called the young remainder.

Example.

1/ Consider the function $f(x) = \sin x$. the Taylor-young's formula up to order 3 in the neighborhood of 0 is

$$\sin x = x - \frac{x^3}{6} + O(x^3).$$

2/ Consider the function e^x. the Taylor-young's formula up to order 4 in the neighborhood of 0 is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^4).$$

Remark:

If $x_0 = 0$, we obtain the Maclaurin-young formula

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + O(x^n).$$

2/ Limited development.

* Limited development in the neighborhood of 0.

Definition.

Let f be a function defined in the neighborhood of $x=0$. We say that f admits a limited development of order n in the neighborhood of 0 if there exist real numbers $a_0, a_1, a_2, \dots a_n$ and a function ε such that for any non-zero element x of an interval I of $\mathbb{R}$:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + x^n \varepsilon(x).$$
$$= P_n(x) + x^n \varepsilon(x).$$

such that $\lim_{x \to 0} \varepsilon(x) = 0$

Remark.

The polynomial $P_n(x)$ is called regular part of the limited development and $x^n \varepsilon(x)$ is remainder or complementary part.

Example

Let $f(x) = \frac{1}{1-x}$, f admits $LD_n(0)$. indeed

we have:

$$
\begin{array}{r|l}
1 & 1 - x \\
\cline{2-2}
1 - x & 1 + x + x^2 + \cdots + x^n \\
\cline{1-1}
x & \\
x - x^2 & \\
\cline{1-1}
x^2 & \\
x^2 - x^3 & \\
\cline{1-1}
x^3 & \\
\vdots & \\
x^n - x^{n+1} & \\
\cline{1-1}
x^{n+1} &
\end{array}
$$

so:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x}.$$

$$= P_n(x) + x^n \cdot \frac{x}{1-x}.$$

with $\varepsilon(x) = \frac{x}{1-x}$ and $\lim_{x \to 0} \varepsilon(x) = 0$.

Then, the function $f(x) = \frac{1}{1-x}$, $x \neq 1$ admits a limited development of order n at $x = 0$.

* properties of Limited development

- If f admits a $LD_n(x_0)$. then $\lim_{x \to x_0} f(x)$ exists, finite and is equal to a_0.

- If $\lim_{x \to x_0} f(x)$ does not exist, then f cannot admit a limited development at x_0.

- A function does not necessarily have an $LD_n(x_0)$ but if it does, it is unique.

- parity.
  - If f is an even function, then the regular part $P_n(x)$ of $LD_n(x_0)$ is an even polynomial.
  - If f is an odd function, then the regular part $P_n(x)$ of $LD_n(x_0)$ is an odd polynomial

- The $LD_n(x_0)$ of a polynomial of degree n is the polynomial itself.

Remark.

* Taylor-young's formula of f of order n at x_0 is $LD_n(x_0)$ where $a_n = \frac{f^{(n)}(x_0)}{n!}$

* If the function f is n-times differentiable at x_0 then it admits a limited development of order n at x_0 by the Taylor-young's formula

Limited development of usual functions.

Below; we show some well-known limited development of common functions of order n at $x=0$

① $f(x) = e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + O(x^n).$$

② $f(x) = \frac{1}{1-x}$

$$f(x) = \frac{1}{1-x} = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + O(x^n)$$

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we have :

$$\begin{cases} f(x) = \dfrac{1}{1-x} \\ f'(x) = \dfrac{1}{(1-x)^2} \\ f''(x) = \dfrac{2}{(1-x)^3} \\ f'''(x) = \dfrac{6}{(1-x)^4} \end{cases} \Longleftrightarrow \begin{cases} f(0) = 1 \\ f'(0) = 1 \\ f''(0) = 2 \\ f'''(0) = 6 \end{cases}$$

So :

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + O(x^n).$$

③ $f(x) = \ln(1+x)$.

$$f(x) = \ln(1+x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + O(x^n).$$

we have:

$$\begin{cases} f(x) = \ln(1+x) \\ f'(x) = \dfrac{1}{1+x} \\ f''(x) = \dfrac{-1}{(1+x)^2} \\ f'''(x) = \dfrac{2}{(1+x)^3} \\ f''''(x) = \dfrac{-6}{(1+x)^4} \end{cases} \Longleftrightarrow \begin{cases} f(0) = 0 \\ f'(0) = 1 \\ f''(0) = -1 \\ f'''(0) = 2 \\ f^{(4)}(0) = -6. \end{cases}$$

So $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots + (-1)^{n+1}\dfrac{x^n}{n} + O(x^n).$

4) $f(x) = \cos x.$

$$f(x) = \cos x = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots \frac{f^{(n)}(0)}{n!}x^n + O(x^{2n+1})$$

we have:

$$\begin{cases} f(x) = \cos x \\ f'(x) = -\sin x \\ f''(x) = -\cos x \\ f'''(x) = \sin x \end{cases} \Longleftrightarrow \begin{cases} f(0) = 1 \\ f'(0) = 0 \\ f''(0) = -1 \\ f'''(0) = 0 \end{cases}$$

so

$$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^{2n+1})$$

5) $f(x) = \sin x.$

$$f(x) = \sin x = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + O(x^{2n+2})$$

we have

$$\begin{cases} f(x) = \sin x \\ f'(x) = \cos x \\ f''(x) = -\sin x \\ f'''(x) = -\cos x \end{cases} \Longleftrightarrow \begin{cases} f(0) = 0 \\ f'(0) = 1 \\ f''(0) = 0 \\ f'''(0) = -1. \end{cases}$$

so

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+2})$$

6) $f(x) = ch(x)$.

$$f(x) = ch(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + O(x^{2n+1})$$

we have:

$$\begin{cases} f(x) = ch(x) = \frac{e^x + e^{-x}}{2} \\ f'(x) = \frac{e^x - e^{-x}}{2} \\ f''(x) = \frac{e^x + e^{-x}}{2} \\ f'''(x) = \frac{e^x - e^{-x}}{2} \end{cases} \Longrightarrow \begin{cases} f(0) = 1 \\ f'(0) = 0 \\ f''(0) = 1 \\ f'''(0) = 0 \end{cases}$$

so

$$ch(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + O(x^{2n+1})$$

7) $f(x) = sh(x)$.

$$f(x) = sh(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + O(x^{2n+2})$$

we have:

$$\begin{cases} f(x) = sh(x) = \frac{e^x - e^{-x}}{2} \\ f'(x) = \frac{e^x + e^{-x}}{2} \\ f''(x) = \frac{e^x - e^{-x}}{2} \\ f'''(x) = \frac{e^x + e^{-x}}{2} \end{cases} \Longrightarrow \begin{cases} f(0) = 0 \\ f'(0) = 1 \\ f''(0) = 0 \\ f'''(0) = 1 \end{cases}$$

so

$$sh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+2}).$$

Operations on Limited development.

Sum

If f admits a $LD_n(0)$: $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + O(x^n)$

and g admits a $LD_n(0)$: $g(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n + O(x^n)$

Then $f+g$ admits a $LD_n(0)$ given by :

$$(f+g)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots + (a_n + b_n)x^n + O(x^n)$$

Example.

Find the $LD_4(0)$ of : $e^x + \ln(1+x)$

we have :

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O(x^4)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^4).$$

so

$$e^x + \ln(1+x) = 1 + 2x + \frac{x^3}{2} - \frac{5x^4}{24} + O(x^4).$$

product.

If f admits a $LD_n(0)$: $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + O(x^n)$

and g admits a $LD_n(0)$: $g(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n + O(x^n)$

Then the product fg admits a $LD_n(0)$, obtained by retaining only the monomials of degree at most n in :

$$(a_0 + a_1 x + \dots + a_n x_n) \times (b_0 + b_1 x + \dots + b_n x^n).$$

Example:

Find the $LD_3(0)$ of $f(x) = \cos x \sin x$.

we have:

$$\cos x = 1 - \frac{x^2}{2} + O(x^3)$$

$$\sin x = x - \frac{x^3}{6} + O(x^3).$$

So

$$\cos x \sin x = x - \frac{2x^3}{3} + O(x^3).$$

Quotient

If f admits a $LD_n(0)$: $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + O(x^n)$

and g also admits a $LD_n(0)$: $g(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n + O(x^n)$

Then $\frac{f}{g}$ admits a $LD_n(0)$ obtained by performing the division according to increasing degrees up to order n of the polynomial $(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)$ by the polynomial $(b_0 + b_1 x + \dots + b_n x^n)$

Example

Find the $LD_5(0)$ of $f = \tan x = \frac{\sin x}{\cos x}$.

we have:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^5).$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^5)$$

So:

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^5)}{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^5)}$$

$$
\begin{array}{l|l}
x - \dfrac{x^3}{6} + \dfrac{x^5}{120} + o(x^5) & 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} + o(x^5) \\
\cline{2-2}
-x + \dfrac{x^3}{2} - \dfrac{x^5}{24} + o(x^5) & x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 \\
\cline{1-1}
\dfrac{x^3}{3} - \dfrac{x^5}{30} + o(x^5) & \\
-\dfrac{x^3}{3} + \dfrac{x^5}{6} + o(x^5) & \\
\cline{1-1}
\dfrac{2x^5}{15} + o(x^5) & \\
-\dfrac{2x^5}{15} + o(x^5) & \\
\cline{1-1}
o(x^5) &
\end{array}
$$

So :

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^5).$$

Composition

If g admits a $LD_n(0)$ and f admits a $LD_n(g(0))$ $(g(0)=0)$,

then $f \circ g$ admits a $LD_n(0)$ obtained by substituting the $LD_n(0)$ of g into that of f and keeping only the monomials of degree n or less.

Example.

Find the $LD_3(0)$ of : $\sin\left(\frac{1}{1-x}-1\right)$.

we have:

- $\frac{1}{1-x}-1 = 1+x+x^2+x^3-1+O(x^3)$.

$$= x+x^2+x^3+O(x^3).$$

- $\sin x = x - \frac{x^3}{6} + O(x^3)$

SO:

$$\sin\left(\frac{1}{1-x}-1\right) = (x+x^2+x^3) - \frac{(x+x^2+x^3)^3}{6} + O(x^3)$$

$$= x+x^2+x^3 - \frac{x^3}{6} + O(x^3) = x+x^2+\frac{5}{6}x^3+O(x^3).$$

Differentiability.

If $f: I \longrightarrow \mathbb{R}$ admits a $LD_{n+1}(0)$ and f is differentiated at least $n+1$ times, then f' admits a $LD_n(0)$ obtained by differentiating the Limit development of f.

Example.

Compute $LD_3(0)$ for $f(x) = \frac{1}{(1-x)^2}$.

since $\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)'$ and

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + O(x^4).$$

SO:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + O(x^3).$$

Remark. we will often work at $x_0 = 0$ based on changes of variables

① If $x_0 \in \mathbb{R}^*$, we put $t = x - x_0$. then $x \longrightarrow x_0$, $t \longrightarrow 0$

② If $x_0 \longrightarrow \infty$ we put $t = \frac{1}{x}$ then $x \longrightarrow \infty$, $t \longrightarrow 0$

Example

Find $LD_n(1)$ for the function e^x.

we put $t = x - 1$ when $x \longrightarrow 1$, $t \longrightarrow 0$

then $x = t + 1$.

So:

$$e^x = e^{t+1} = e \cdot e^t = e\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + o(t^n)\right)$$

$$= e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \cdots + \frac{(x-1)^n}{n!} + o\left((x-1)^n\right)\right)$$

Generalized LD.

Let f be a function defined in the neighborhood of 0 except possibly at 0.

Suppose that f does not admit a $LD_n(0)$ but the function $x^\alpha f(x)$ $(\alpha > 0)$ admits a $LD_n(0)$

$$x^\alpha f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n).$$

Hence

$$f(x) = \frac{1}{x^\alpha}\left(a_0 + a_1 x + \cdots + a_n x^n + o(x^n)\right).$$

This expression is called the generalized Limited development in the neighborhood of 0.

Example

Consider the function $f(x) = \frac{1}{x-x^2}$.

- we have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x-x^2} = +\infty$ so f does not admit a $LD(0)$.

- But

$$x f(x) = x \cdot \frac{1}{x-x^2} = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + O(x^n).$$

- The generalized limited development of f is:

$$f(x) = \frac{1}{x}\left(1 + x + x^2 + \cdots + x^n + O(x^n)\right)$$

$$= \frac{1}{x} + 1 + x + x^2 + \cdots + x^{n-1} + O(x^{n-1})$$

Applications on calculating limits

Limited development is very useful in the case of an inderminate form when computing a limit

① Calculate: $\lim_{x \to 0} \frac{x - \sin x}{x^3} = \frac{0}{0}$ (IF)

we have:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^5).$$

we obtain

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^5)\right)}{x^3}$$

$$= \lim_{x \to 0} \frac{1}{6} - \frac{x^2}{120} + O(x^2)$$

$$= \frac{1}{6}.$$

② Calculate $\lim_{x \to 0} \frac{\tan x - x}{x^3} = \frac{0}{0}$ (IF)

we have:

$$\tan x = x + \frac{x^3}{3} + O(x^3)$$

so:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{x + \frac{x^3}{3} - x + O(x^3)}{x^3} = \lim_{x \to 0} \frac{1}{3} + O(x^3)$$

$$= \frac{1}{3}.$$